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FUNCTIONAL ANALYSIS

A Course of Lectures, Individual Assignments, and Methodological Guidelines for Master's Degree Students of the Educational Program "Secondary Education (Mathematics)"

(for International Students)

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CONTENTS

INTRODUCTION. BRIEF HISTORICAL INFORMATION

Gilbert David (1862-1943)
German mathematician

Functional analysis is a branch of modern mathematics that developed at the turn of the nineteenth and twentieth centuries at the intersection of mathematical analysis, linear algebra, geometry, and the theory of integral equations. Its formation as an independent mathematical discipline was closely connected with the study of infinite-dimensional spaces, linear operators, functionals, integral equations, and related problems of mathematical analysis.

An important stage in the development of functional analysis was the systematic study of infinite-dimensional normed spaces and linear operators acting on them. The work of mathematicians such as David Hilbert, Maurice Fréchet, Stefan Banach, Hans Hahn, and others played a fundamental role in establishing the foundations of the subject. The development of functional analysis was particularly associated with the emergence of the theory of normed and complete normed spaces. Complete normed spaces are now known as **Banach spaces**, while complete inner-product spaces are known as **Hilbert spaces**. These spaces constitute fundamental objects of modern functional analysis.

Stefan Banach

The development of functional analysis was also closely connected with the **Lviv School of Mathematics**, whose leading figures included Stefan Banach and Hugo Steinhaus. According to the historical materials of the Ivan Franko National University of Lviv, one of the major achievements of the Lviv mathematical school was the establishment of the foundations of functional analysis.

Historical reference: Stefan Banach

Stefan Banach (30 March 1892 – 31 August 1945) was a Polish mathematician and one of the founders of modern functional analysis. He was one of the leading figures of the Lviv School of Mathematics and made fundamental contributions to the development of functional analysis and several other areas of mathematics.

Banach was born in Kraków. In June 1910, he passed his secondary-school examination and subsequently moved to Lviv, where he studied at the Lviv Polytechnic. He initially enrolled in the Faculty of Machine Construction and later transferred to the Faculty of Engineering. Because of financial difficulties, he supported himself, apparently through private tutoring, and did not complete a full university degree at the Lviv Polytechnic.

In 1920, Professor Antoni Łomnicki invited Banach to work as an assistant at the Department of Mathematics of the Lviv Polytechnic. In the same year, Banach submitted his doctoral dissertation, "On Operations on Abstract Sets and Their Applications to Integral Equations." The dissertation was accepted by the Faculty of Philosophy of the University of Lviv, and he passed the required examinations with distinction. The formal ceremony at which Banach received his doctoral degree took place on 22 January 1921.

In 1922, Banach completed his habilitation at the University of Lviv and became an extraordinary professor. In 1927, he was appointed an ordinary professor. Thus, the University of Lviv recognized his exceptional mathematical achievements despite the fact that he had not completed a conventional university degree.

Banach became a central figure of the Lviv School of Mathematics. The school included a number of outstanding mathematicians, among them Hugo Steinhaus, Stanisław Mazur, Władysław Orlicz, Stefan Kaczmarz, Juliusz Schauder, Władysław Nikliborc, Herman Auerbach, and others. The representatives of the school made significant contributions to twentieth-century mathematics and became internationally recognized.

In 1924, Banach became a corresponding member of the Polish Academy of Sciences. In 1928, together with Hugo Steinhaus, he founded the mathematical journal Studia Mathematica; its first issue appeared in 1929. The journal became an important publication venue for research in functional analysis and related areas.

Banach's most influential monograph, Théorie des opérations linéaires, was published in 1932. A Polish edition had appeared in 1931, while a Ukrainian translation, prepared by Professor Myron Zarytskyi, was published in 1948 under the title Курс функціонального аналізу. This work played an important role in the development and dissemination of the theory of linear operators and functional analysis.

Banach's scientific achievements were not limited to functional analysis. He also made important contributions to the theory of functions, orthogonal series, measure theory, and set theory. The official historical materials of the Ivan Franko National University of Lviv record 58 publications by Banach, including six published posthumously.

During the German occupation of Lviv, Banach worked at the research institute of Professor Rudolf Weigl, where research and vaccine production against typhus were carried out. This work helped Banach survive the occupation. After the University of Lviv resumed its activities in 1944, he renewed his scientific contacts with mathematicians in the Soviet Union and Poland. Stefan Banach died in Lviv on 31 August 1945.

The Lviv School of Mathematics

The **Lviv School of Mathematics** was a group of mathematicians who worked in Lviv during the 1920s and 1930s. They were associated primarily with the University of Lviv, the Lviv Polytechnic, and other educational institutions of the city.

The undisputed leaders of the school were **Stefan Banach and Hugo Steinhaus**. Among its representatives were mathematicians who later became internationally recognized for their contributions to functional analysis, probability theory, set theory, measure theory, topology, and other branches of mathematics.

One of the most important achievements of the Lviv mathematical school was the establishment of the foundations of functional analysis. In particular, the research carried out by Banach and his collaborators contributed to the systematic development of the theory of infinite-dimensional spaces and linear operators.

An important part of the mathematical life of interwar Lviv was the tradition of regular mathematical meetings and discussions. The famous **Scottish Café** became one of the meeting places of Lviv mathematicians. Mathematical problems discussed by members of the school were recorded in the well-known **Scottish Book**, which has become an important historical document of twentieth-century mathematics.

METRIC SPACES

The concept of a limit is a basic concept of mathematical analysis, with the help of which the concept of derivative, integral, etc. is introduced. The concept of limit uses the concept of distance between numbers. Indeed, the fact that the numerical sequence $\{x_n\} \to a$ means that $|x_n - a| \to 0$. In order to introduce the concept of a limit for more complex objects than a number, it is necessary to define the concept of distance between abstract objects, which leads to the concept of metric space.

Definition. A set of objects of an arbitrary nature $\{x,y,z,\ldots\}$ is called a *metric space* if the concept of distance $\rho(x,y)$, is defined on this set, which matches any two elements x and y with the number $\rho(x,y)$, while the following three axioms are fulfilled:

1. $\rho(x,y)\geq 0$, $\rho(x,y)=0 \Leftrightarrow x\equiv y$ – *axiom of reflexivity;*
2. $\rho(x,y)= \rho(y,x)$ – *symmetry axiom;*
3. $\rho(x,y)\leq \rho(x,z)+ \rho(z,y)$ – *triangle inequality.*

A metric space with a set of elements X and given distance ρ is denoted by (X, ρ).

Examples of metric spaces.

1. Space R^1, X – elements are real numbers, the distance is defined as follows:

$\rho(x,y)=|x-y|$.

Let's check whether this space will be metric. For this, three metric axioms must be fulfilled.

a) $\rho(x,y)\geq 0$ – in power of properties of the modulus of a real number. If $\rho(x,y)=0 \Rightarrow |x-y|=0 \Rightarrow x\equiv y$. If $x\equiv y$, then $\rho(x,y)= \rho(y,y)=|y-y|=0$.

The first axiom was proved. Then we have:

б) $\rho(x,y) = |x-y| = |(-1)(y-x)| = |-1||y-x| = |y-x| = \rho(y,x)$.

в) $\rho(x,y) = |x-y| = |x-z+z-y| \le |x-z| + |z-y| = \rho(x,z) + \rho(z,y)$.

Since all three axioms from the definition of a metric space are fulfilled, then this space is metric

2. Space of isolated points. The set $\{x,y,z,...\}$ consists of elements of an arbitrary nature,

$$\rho(x,y) = \begin{cases} 1, & x \ne y, \\ 0, & x = y. \end{cases}$$

Check the implementation of the distance axioms yourself.

3. Space R^n, $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)$, the elements of vectors are real numbers, $\rho(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Let's check the three metric axioms:

a) $\rho(x,y) \ge 0$ – follows from the properties of the square of a real number, the sum, and the square root of real numbers.

If $x=y$, then $\rho(x,y)=\rho(x,x)=\sqrt{\sum_{i=1}^{n}(x_i - x_i)^2}=0$. Let $\rho(x,y)=0$, then $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}=0$. From this it follows that $x_i = y_i$ ($i = \overline{1,n}$),and this means that $x=y$. The first axiom is proved.

b) $\rho(x,y)=\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}((-1)(y_i - x_i))^2} = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = \rho(y,x)$.

c) Let's prove the triangle inequality using the Cauchy-Bunyakovsky inequality:

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}.$$

Consider the square of the distance $\rho(x,y)$:

$$(\rho(x,y))^2 = \sum_{i=1}^{n}(x_i - y_i)^2 = \sum_{i=1}^{n}(x_i - z_i + z_i - y_i)^2 = \sum_{i=1}^{n}(x_i - z_i)^2 + 2\cdot\sum_{i=1}^{n}(x_i - z_i)(z_i - y_i) +$$

$+\sum_{i=1}^{n}\left(z_i-y_i\right)^2 \le$ (we apply the Cauchy-Bunyakovsky inequality to the second term) $\le \sum_{i=1}^{n}\left(x_i-z_i\right)^2+2\cdot\sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}\cdot\sqrt{\sum_{i=1}^{n}(z_i-y_i)^2}+\sum_{i=1}^{n}\left(z_i-y_i\right)^2=$ $=(\rho(x,z)+\rho(z,y))^2$.

So, $(\rho(x,y))^2\le(\rho(x,z)+\rho(z,y))^2$. Since the right and left sides of the inequality are non-negative, we can remove the square, then we get the triangle inequality.

4. Space R_0^n, $x=(x_1,x_2,\ldots,x_n)$, $y=(y_1,y_2,\ldots,y_n)$,), the elements of vectors are real numbers, $\rho(x,y)=\max\limits_i\left|x_i-y_i\right|$.

Let's prove the third axiom:

$$\left|x_i-y_i\right|=\left|x_i-z_i+z_i-y_i\right|\le\left|x_i-z_i\right|+\left|z_i-y_i\right|\le\max_i\left|x_i-z_i\right|+\max_i\left|z_i-y_i\right|$$

This inequality holds for any i, and therefore also for the maximum. We will have: $\max\limits_i\left|x_i-y_i\right|\le\max\limits_i\left|x_i-z_i\right|+\max\limits_i\left|z_i-y_i\right|$.

5. Space R_1^n, $x=(x_1,x_2,\ldots,x_n)$, $y=(y_1,y_2,\ldots,y_n)$,), the elements of vectors are real numbers, $\rho(x,y)=\sum_{i=1}^{n}\left|x_i-y_i\right|$. Axioms are proved similarly to previous cases. We offer readers to prove them on their own.

6. Space l_2, the elements of which are infinite sequences of the species $x=(x_1,x_2,\ldots,x_n,\ldots)$ and the condition is met $\sum_{i=1}^{\infty}x_i^2<\infty$, the elements of sequences are real numbers. The distance is set as follows: $\rho(x,y)=\sqrt{\sum_{i=1}^{\infty}\left(x_i-y_i\right)^2}$.

Let's check the correctness of the entered metric, namely, make sure that the series appearing under the sign of the radical is convergent.

We have that: $(x_i-y_i)^2\le 2\cdot(x_i^2+y_i^2)$ and that series with common terms x_n^2 and y_n^2 are convergent. Then the sum of these series also converges. It follows that the series with the common term in the left-hand side of the last inequality ($(x_n - y_n)^2$) also coincides. The distance axioms are tested in the same way as it was done for the space R^n.

7. Space $C_{[a,b]}$ – the space of continuous functions on a segment $[a,b]$ whose element is $x=x(t)$.The distance is entered as follows:

$$\rho(x,y) = \max_t |x(t) - y(t)|$$

Let's prove the third axiom. For any $t \in [a,b]$ we have: $|x(t)-y(t)|$ = $|x(t)-z(t)+z(t)-y(t)| \le |x(t)-z(t)|+|z(t)-y(t)| \le$

$\le \max_t |x(t)-z(t)| + \max_t |z(t)-y(t)| = \rho(x,z)+\rho(z,y)$. Again, if such an inequality holds for any $t \in [a,b]$, then it will be performed for $\max_t$, that is: $\max_t |x(t)-y(t)| \le \rho(x,z)+\rho(z,y)$, thereby proving the triangle inequality.

8. Space C_L – space of continuous $[a,b]$ functions with distance $\rho(x,y) = \int_a^b |x(t)-y(t)| dt$.

Let's prove the first axiom. Due to the properties of the modulus of the number and the integral - $\rho(x,y) \ge 0$. Let $x(t)=y(t)$, then $\rho(x,y) = \rho(x,x) = \int_a^b 0dt$ =0. If $\rho(x,y)$=0, then we reason as follows: let there be a point t_0 in which $x(t_0) \ne y(t_0)$ and $x(t_0)>y(t_0)$, then due to the continuity of functions $x(t)$ i $y(t)$, $x(t)>y(t)$ in some neighborhood t_0. Then we will have that $\int_a^b |x(t)-y(t)| dt$ >0. We have come to a contradiction. It follows that $x(t)$ should be equal to $y(t)$. The second and third axioms are proved similarly to the previous cases.

9. Space C_{L_2} – space of continuous functions on $[a,b]$, whose metric is entered as follows:

$$\rho(x,y)=\sqrt{\int_a^b (x(t)-y(t))^2 dt}\,.$$

Let's prove the triangle inequality. For this, we will use the Cauchy-Bunyakovsky integral inequality:

$$\left|\int_a^b f(t)\cdot g(t)dt\right| \le \sqrt{\int_a^b f^2(t)dt}\cdot\sqrt{\int_a^b g^2(t)dt}\,.$$

We have:

$$\rho(x,y)^2=\int_a^b (x(t)-z(t))^2 dt+2\int_a^b (x(t)-z(t))(z(t)-y(t))dt+\int_a^b (z(t)-y(t))^2 dt \le$$

$$\le \int_a^b (x(t)-z(t))^2 dt+2\cdot\sqrt{\int_a^b (x(t)-z(t))^2 dt}\cdot\sqrt{\int_a^b (z(t)-y(t))^2 dt}+\int_a^b (z(t)-y(t))^2 dt=$$

$$=\rho(x,z)^2+2\rho(x,z)\rho(z,y)+\rho(z,y)^2=(\rho(x,z)+\rho(z,y))^2.$$

Taking the square root of negative numbers, we get the triangle inequality.

10. Space C_0 – the elements of which are infinite sequences that converge to zero: $x=(x_1,x_2,\dots,x_n,\dots)$, $\lim\limits_{n\to\infty} x_n \to 0$, $\rho(x,y)=\sup\limits_i |x_i-y_i|$. This definition of the distance is correct, because $|x_i-y_i|\le|x_i|+|-y_i|$. And since the sequences $\{x_i\}$ and $\{y_i\}$ converge to zero, they are bounded by an arbitrary number. Then $|x_i-y_i|$, as the magnitude is bounded, has a supremum. The fulfillment of the distance axioms is obvious.

11. Space C_m . The elements of the space are arbitrary bounded sequences of real numbers, $\rho(x,y)=\sup\limits_i|x_i-y_i|$, $|x_m|<m$. We suggest that the reader independently check the fulfillment of the metric axioms and the correctness of such a distance definition

Control questions.

1. Give the definition of a metric space.
2. Define metric spaces of isolated points, R^0, l_2, $C_{[a,b]}$, C_l, C_{l_2}.

3. Write the Cauchy-Buniakovsky inequality in integral form and indicate the scheme of its proof.
4. Is it possible in space $C_{[a,b]}$ determine the distance using the formula $\rho(x,y)=\min_t |x(t)-y(t)|$.
5. What metric spaces are studied in the course of high school mathematics, the course of linear algebra, mathematical analysis, the theory of Fourier series.

Exercises.

1. Check the metric axioms for the space of isolated points.
2. Given $\rho(x,y)=\sin^2|x-y|$, $x,y\in R$, will it be $\rho(x,y)$ metric.
3. Given the space $C_{[a,b]}$. Is it possible to enter the distance according to the formula $\rho(x,y)=\min_t |x(t)-y(t)|$?
4. Given $x\in R$, $\rho(x,y)=\mathrm{arctg}|x-y|$. Will it be $\rho(x,y)$ metric.

The solution.

1. Given $\rho(x,y)=\begin{cases}0, \text{ якщо } x=y\\ 1, \text{ якщо } x\neq y\end{cases}$.

The fulfillment of the first and second metric axioms is obvious. Let's check whether the triangle axiom holds $\rho(x,y)\le\rho(x,z)+\rho(z,y)$.

I will assume the opposite, that this inequality does not hold. This is possible when there is a unit on the left side $(x\neq y)$, and in the right part - zero, that is $x=z$, $z=y$. It follows that $x=y$. And this contradicts the assumption that $x\neq y$.

2. No, it won't, because the first axiom of the metric doesn't hold. Namely: nothing follows from the fact that the distance is zero $x=y$. (From equality $\sin^2|x-y|=0 \Rightarrow \sin|x-y|=0 \Rightarrow (x-y)=k\pi,\ k\in Z$, $x-y=\pm k\pi$, $k\in Z$, $x=\pm k\pi$).

3. No, because the first axiom of the metric will not be fulfilled. Indeed, by taking two continuous functions $x(t)$ and $y(t)$, which intersect, for example, at one point we will have that $\rho(x,y)=0$. However, these

functions do not coincide identically.

4. The fulfillment of the first and second distance axioms is obvious. Let's prove the triangle inequality:

$$\rho(x,y) = \mathrm{arctg}\,|x-y| = \mathrm{arctg}\,|x-z+z-y| \le \mathrm{arctg}\,(|x-z|+|z-y|) \le$$
$$\le \mathrm{arctg}\,|x-z| + \mathrm{arctg}\,|z-y| = \rho(x,z)+\rho(z,y)$$

(we propose to prove the penultimate inequality independently).

Problems.

1. Verify the metric axioms for space R_0^n, R_1^n, C_0, C_1, C_m.
2. $\rho(x,y)-$ metric. Prove that $\ln(1+\rho(x,y))$ also a metric.
3. Given that $\rho(x,y)$ – metric. Given that $\rho_1(x,y)=\dfrac{\rho(x,y)}{1+\rho(x,y)}$ also a metric.
4. Given: $x \in R$, $\rho(x,y) = \left| e^x - e^y \right|$. Check the metric axioms.
5. Given a set of differentiable functions *x(t),* $\rho(x,y) = \int\limits_b^a |x(t)-y(t)|dt + \max\limits_t |x'(t)-y'(t)|$. Check the metric axioms.
6. Given $x \in R$, $\rho(x,y)=\sqrt{|x-y|}$. Check the fulfillment of the metric axioms.
7. Given $x \in R$, $\rho(x,y)=\min\{1,|x-y|\}$. Check the fulfillment of the metric axioms.
8. Given $x \in R$, $\rho(x,y)=\left|x^3-y^3\right|$. Will it be $\rho(x,y)$ metric.
9. Given $x\in[0,\infty)$, $\rho(x,y)=\left|x^2-y^2\right|$. Will it be $\rho(x,y)$ metric.
10. Given $x \in l_2$, $\rho(x,y)=\sum\limits_{n=1}^{\infty}\dfrac{|x_n-y_n|}{n}$. Will it be $\rho(x,y)$ metric.

SEQUENCES IN METRIC SPACES. CONVERGENCE

Let us have a metric space (X,ρ).

<u>Definition.</u> If to each $n \in N$ taken in ascending order, the element is matched $x_n \in (X,\rho)$, then it is said that in this space *the sequence* $\{x_n\}$ *is given*.

<u>Examples.</u>

$R^1 : \left\{\frac{1}{n}\right\};$

$R^m : \left\{x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}\right\};$

$C_{[a,b]} : \left\{x^n\right\}, x \in [0,1].$

Here, m is the dimension of the space, n is the general term of the sequence.

<u>Definition.</u> Element a, which belongs to space (X,ρ) is called *the sequence limit* $\{x_n\}$, if for $\forall \varepsilon > 0\ \exists N$, that as soon as $n \geq N$, then $\rho(x_n, a) < \varepsilon$.

<u>Theorem 1.</u> If the sequence has a limit, then it is unique.

<u>Proof.</u> Suppose that there are two different limits a and b then

$\forall \varepsilon > 0\ \exists N_1,\ n \geq N_1 : \rho(x_n, a) < \varepsilon;$

$\forall \varepsilon > 0\ \exists N_2,\ n \geq N_2 : \rho(x_n, b) < \varepsilon;$

Let $N=\max\{N_1,N_2\}$, if $n > N$, then both inequalities are fulfilled simultaneously. Then we have: $\rho(a,b) \leq \rho(a,x_n) + \rho(x_n,b) \leq 2\varepsilon$, and this means that $\rho(a,b)=0$. So $a=b$.

The theorem is proved.

<u>Theorem 2.</u> If the sequence $\{x_n\}$ in the metric space has a limit, then this sequence is bounded.

<u>Proof.</u> Indeed, if the sequence has a limit, then this means that

$\rho(x_n,a)\to 0,\ n\to\infty$. But $\rho(x_n,a)$ – is a numerical sequence and it has a limit equal to zero, so it is bounded, that is, it exists *M*, that $\rho(x_n,a)\le M$, which means boundedness, because all elements of the sequence are inside a sphere centered at a point *a* and radius *M*.

Definition. The sequence $\{x_n\}$ from the metric space (X,ρ) is called a *fundamental* or *Cauchy sequence* or *self-convergent* if for $\forall\varepsilon>0\ \exists N$, that as soon as $n,m>$N, then $\rho(x_n,x_m)<\varepsilon$.

Theorem 3. If a sequence in a metric space has a limit, then any of its subsequences also has a limit.

Theorem 4. If the sequence has a limit, then it is fundamental.

Proof. For $\forall\,\varepsilon>0\ \exists N$, that at $n\ge N:\rho(x_n,a)<\varepsilon$, and let $m\ge N\Rightarrow\rho(x_m,a)<\varepsilon$

Let's evaluate, $\rho(x_n,x_m)\le\rho(x_n,a)+\rho(a,x_m)\le\varepsilon+\varepsilon=2\varepsilon$, and this is a condition of fundamentality.

The theorem is proved.

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Remark. The converse statement is false, although it is true for numerical sequences.

Convergence in metric spaces.

Convergence of sequences in metric spaces is also called distance convergence (if $x_n\to a$ at $n\to\infty$, then the distance between these elements goes to zero, $\rho(x_n,a)\to 0$). In each specific space, this convergence can be characterized more meaningfully.

Theorem 5. In order to sequence $\{x_1^{(n)},x_2^{(n)},\ldots,x_m^{(n)}\}\in R^m$ headed for the limit element $a=(a_1,a_2,\ldots,a_m)$ it is necessary and sufficient that the numerical sequences of the coordinates of this sequence go to the corresponding coordinates of the element *a*.

Proof. Necessity. It is given that the sequence matches to the element

a by distance. This means that for $\forall\varepsilon > 0 \ \exists N, \ n \geq N,$ than

$\rho(x_n, a) = \sqrt{\left(x_1^{(n)} - a_1\right)^2 + \left(x_2^{(n)} - a_2\right)^2 + \ldots + \left(x_m^{(n)} - a_m\right)^2} < \varepsilon$. From here follows:

$$\left|x_1^{(n)} - a_1\right| < \varepsilon,$$
$$\left|x_2^{(n)} - a_2\right| < \varepsilon,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\left|x_m^{(n)} - a_m\right| < \varepsilon.$$

Note that we obtain the last inequalities by reasoning from the opposite. If, for example, the first expression were bigger ε, then in the square it would be bigger ε^2, would get a contradiction. The last inequalities mean that:

$$\lim_{n\to\infty} x_1^{(n)} = a_1, \ldots, \lim_{n\to\infty} x_m^{(n)} = a_m \ .$$

The necessity is proven.

Sufficiency. Let the numerical sequences of coordinates have limits. Then $\forall\varepsilon > 0$:

$\exists N_1, n > N_1,$ то $\left|x_1^{(n)} - a_1\right| < \varepsilon$;

$\exists N_2, n > N_2,$ то $\left|x_2^{(n)} - a_2\right| < \varepsilon$;

…. …. …. …. …. …. ….

$\exists N_m, n > N_m,$ то $\left|x_m^{(n)} - a_m\right| < \varepsilon$;

Let's take $N=\max\{N_1, N_2, \ldots, N_m\}$, whereas as soon as $n>N$, then all inequalities will be fulfilled simultaneously. We will have: $\rho(x_n, a) = \sqrt{\left(x_1^{(n)} - a_1\right)^2 + \ldots + \left(x_m^{(n)} - a_m\right)^2} < \sqrt{\varepsilon^2 + \varepsilon^2 + \ldots + \varepsilon^2} = \varepsilon\sqrt{m}$, $\sqrt{m}$ – constant.

Due to arbitrary ε, this means that the distance goes to zero, which proves sufficiency.

The theorem is proved.

In space R^m we introduced the following three metrics:

$$\rho(x,y)=\sqrt{\sum_{i=1}^{m}(x_i-y_i)^2};$$

$$R_0^m:\rho(x,y)=\max_i|x_i-y_i|;$$

$$R_1^m:\rho(x,y)=\sum_{i=1}^{m}|x_i-y_i|.$$

<u>Theorem 6.</u> In space R^m all three metrics are equivalent in the sense that the convergence of the sequence by one metric implies the convergence by the other two metrics.

<u>Proof.</u> Consider the sequence in space R^m $\{x_1^{(n)},x_2^{(n)},...,x_m^{(n)}\}$ *and let it match by metric* $\rho(x,y)=\sqrt{\sum_{i=1}^{m}(x_i-y_i)^2}$. Let's show the convergence according to two other metrics: $\forall\varepsilon>0$ $\exists N,\ n\geq N:\rho(x_n,a)=\sqrt{\sum_{i=1}^{m}(x_i^{(n)}-a_i)^2}<\varepsilon$, whence it follows that: $\sum_{i=1}^{m}(x_i^{(n)}-a_i)^2<\varepsilon^2$; $(x_i^{(n)}-a_i)^2\leq\varepsilon^2$, $\left|x_i^{(n)}-a_i\right|\leq\varepsilon$; $i=\overline{1,m}$. The last inequality holds for any i, therefore also for the maximum: $\max_i\left|x_i^{(n)}-a_i\right|<\varepsilon$ (we got a metric in space R_0^m). We also have $\sum_{i=1}^{m}\left|x_i^{(n)}-a_i\right|<\varepsilon+\varepsilon+\ldots+\varepsilon=m\varepsilon$, which proves the convergence of the sequence in space R_1^m. Two other cases are proved similarly.

The theorem is proved.

<u>Theorem 7.</u> In space l_2 convergence of the sequence by distance implies convergence by coordinates.

<u>Proof.</u> Let $a=(a_1,a_2,a_3,\ldots,a_m,\ldots)$, $\sum_{i=1}^{\infty}a_i^2<\infty$. Let us have a sequence $x_n=\{(x_1^{(n)},x_2^{(n)},\ldots,x_m^{(n)},\ldots)\}$ and let this sequence converge in distance:

$$\forall\varepsilon>0\ \exists N,\ n>N:\rho(x_n,a)=\sqrt{\sum_{i=1}^{\infty}(x_i^{(n)}-a_i)^2}<\varepsilon,$$

than $\sum_{i=1}^{\infty}(x_i^{(n)}-a_i)^2<\varepsilon^2$; $\left|x_i^{(n)}-a_i\right|<\varepsilon$ for any i. The last inequality means that the numerical sequences of the coordinates coincide with the corresponding coordinates of the element a.

Remark. The inverse statement is not always true, i.e. convergence by coordinates does not always result in convergence by distance.

Theorem 8. In space $C_{[a,b]}$ convergence by distance is equivalent to uniform convergence.

Proof. Let the sequence $\{x_n(t)\}\to x_0(t)$. According to the metric of this space, this means that $\forall\varepsilon>0\ \exists N,\quad n\ge N$: $\rho(x_n,x_0)=\max\limits_t\left|x_n(t)-x_0(t)\right|<\varepsilon$. Since the written inequality is fulfilled at $\max\limits_t$, then it will be fulfilled $\forall t\in[a,b]$, that is, the condition will be fair $\left|x_n(t)-x_0(t)\right|<\varepsilon$, and this is a condition of uniform convergence.

Now let the sequence of functions converge to the limit function uniformly. This means that $\forall\varepsilon>0\ \exists N,\ n\ge N$, then $\left|x_n(t)-x_0(t)\right|<\varepsilon$, $\forall t\in[a,b]$, then it will also be fulfilled for $\max\limits_t\left|x_n(t)-x_0(t)\right|<\varepsilon$.

Neighborhoods of points in metric space and related concepts.

Let us have a metric space (X,ρ).

Definition 1. An *open sphere* with the center at a point x_0 and radius R is called the set of points of the metric space that satisfies the inequality $\rho(x,x_0)<R$. If there is also equality $\rho(x,x_0)\le R$, then the sphere is closed.

Example. For space R^3 – it will be the interior of the sphere, indeed $\sqrt{(x_1-x_1^0)^2+(x_2-x_2^0)^2+(x_3-x_3^0)^2}<R$, $x_0=(x_1{}^0,x_2{}^0,x_3{}^0)$ – center of the sphere, $x=(x_1,x_2,x_3)$ – running coordinates. For R^2 the sphere will be the interior of the circle of radius R. For R^1 the sphere will be the interval. For space $C_{[a,b]}$ the open sphere will be the stripe.

Definition 2. *ε – neighborhood* the point x_0 is called an open sphere centered at a point x_0 and radius ε.

Definition 3. Point x_0 is called *the point of contact* of the set E, if in any of its neighborhoods there is at least one point of the set.

Definition 4. The set of all points of contact is called the closure of the set E, and is denoted by $[E]$.

Definition 5. Point x_0 is called an isolated point of the set E if there is a neighborhood of this point that does not contain any point of the set except the given one.

Definition 6. Point x_0 is called a *limit point* for the set E if there are an infinite number of points of the set in any of its neighborhoods E.

Definition 7. Point x_0 is called an *interior point* of the set E if it belongs to the set E together with some of its surroundings.

Definition 8. A set E is called *open* if all its points are interior.

Definition 9. A set E is called *closed* if it contains all its limit points.

Definition 10. The set CE is called the *complement* of the set E to the entire space if all its points belong to the space and do not belong to the set E.

Definition 11. A set A is said to be *dense* in a set B if the closure of A includes B ($[A] \supset B$).

Definition 12. A set A is called *dense everywhere* if its closure coincides with the entire space.

If the set A is dense in the entire space, then any element of the space can be approximated as accurately as we want by the elements of the set A.

Definition 13. A metric space is called *separable* if it contains a countable everywhere dense subset.

Space R^n – separable. A dense subset of this space will obviously be all possible sets of n-rational numbers.

In the theory of approximations of functions, the Weierstrass theorem is well known that any continuous function $f(x)$ on the segment $[a,b]$ can be approximated with any predefined accuracy by an algebraic polynomial of the form:

$P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$.

That is to say,

$\forall \varepsilon > 0 \ \exists P_n(x): \max_x |f(x) - P_n(x)| < \varepsilon$.

A dense subset in space $C_{[a,b]}$ will be a set of algebraic polynomials with rational coefficients. This set is countable.

Theorem 9. In order for the set E to be open, it is necessary and sufficient that its complement is closed.

Proof. Necessity Given a set E, which is open, we prove that CE is closed, i.e it contains all its boundary points. Let the point x be the limit point of the complement. We will prove that it belongs to the complement. This point cannot belong to the set E, because if it did, it would be together with some of its surroundings (since E is open). And this cannot be, because in any neighborhood of x there must be an infinite number of points of complement to E. The necessity is proved.

Sufficiency. It is given that CE is closed. We will prove that E is open, that is, all its points are interior. Let x be some point of the set E, it cannot be a limit point of the complement (why?), so there is a neighborhoodt that does not contain any point of the complement, then this neighborhood belongs to the set E.

The theorem is proved.

Control questions.

1. Give the definition of the limit point, point of contact, closure of the set.
2. Define open and closed set.
3. Give the definition of dense set and separable space.
4. Give the definition of the limit of a sequence in metric space, compare this definition with the definition of the limit of numerical

sequence.

5. Characterize the convergence of sequences in space R^n, l_2, $C_{[a,b]}$.

Exercises.

1. Given a set $\{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-1}{n}, \ldots\}$. Find the limit points, points of contact, limit and closure of the given set.
2. Construct a dense subset in space l_2.
3. Prove the separability of space $C_{[a,b]}$.
4. Prove that from the existence of the limit of a sequence in an arbitrary space, its limitation in this space follows.

The solution.

1. The limit point of this set will be point 1, because in any neighborhood of it there is an infinite number of points of this set. The points of contact will be all the points of the set and point 1. These points will make up the closure and the limit of the set.

2. Such a subset will be the set of elements from the space l_2, in which a finite number of the first coordinates are different from 0, and the other coordinates are zero. Indeed, every element is from space l_2 in this case, we can arbitrarily accurately approximate by an element from a dense subset l_2^*:

$$x=\{x_1, x_2, x_3, \ldots, x_n \ldots\} \in l_2, \ \sum_{i=1}^{\infty} x_k^2 < \infty, \ x^* = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in l_2,$$

$$\rho(x, x^*) = \sqrt{(x_1 - x_1^*) + (x_2 - x_2^*) + \ldots + (x_n - x_n^*) + (x_{n+1} - 0) + (x_{n+2} - 0) + \ldots} \ .$$

Since the series $\sum_{k=1}^{\infty} x_k^2$ is convergent, then its remainder can be made as small as desired.

3. To prove the separability of space $C_{[a,b]}$ it suffices to show that in this space there exists a dense subset closed everywhere. According to

the Weierstrass theorem for any continuous function $x(t)$, which is continuous on the segment $[a,b]$ there is an algebraic polynomial of degree n that approximates this function as accurately as desired. Then it is clear that it is possible to choose an algebraic polynomial with rational coordinates that will approximate the function as accurately as desired $x(t)$. There will be a countable set of such polynomials. And this proves the separability of space $C_{[a,b]}$.

4. We have that sequence $\{x_n\}$ has a limit in an arbitrary metric space (X,ρ). This means that for any $\varepsilon > 0$, there exists N such that at $n \geq N$ the inequality holds $f(x_n,a)<\varepsilon$, that is $f(x_n,a) \to 0$, when $n \to \infty$, but a numerical sequence that has a limit is limited, that is, there is a number $N > 0$, what $\rho(x_n,a) \leq N$. It was found that all members of the sequence are located in an abstract sphere with the center at the point a and radius N, which proves the boundedness of this sequence.

Problems.

1. Given a set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$. Find the limit points, points of contact and closure of the given set.
2. Give examples of dense sets everywhere, construct a dense subset in space l_2.
3. Prove the separability of space R^n, $C_{[a,b]}$.
4. Prove that in order for the set E to be open, it is necessary and sufficient that its complement to the entire space is closed.
5. Prove that the convergence by distance in spaces R_0^n, $R_1^n -$ is equivalent to convergence by coordinates.
6. Prove that a sequence that has a limit in an arbitrary space is bounded.

7. Prove that the existence of the boundary of a sequence in an arbitrary space implies its fundamentality.
8. Prove that three metrics are in spaces R_0^n, R_1^n, R^n are equivalent to each other.
9. Write the neighborhood of points in spaces R^3, R^2, R^1, $C_{[a,b]}$ and depict them in a picture.
10. Construct an abstract sphere with a larger radius, which is contained in the middle of a sphere with a smaller radius.
11. Prove that in order for point X to be a limit point, it is necessary and sufficient for there to be a sequence X_n that would go to X.
12. Given $M=\sup\limits_{x\in E}\{x\}$. Prove that M – the limit point of the set E.
13. Prove that the sequence $U_n(x)=\left\{\dfrac{nx+x^2+n^2}{x^2+n^2}\right\}$ evenly coincides with $U(x)=1$ on the segment $[0,1]$. Will it coincide evenly along the entire numerical axis.
14. Check for uniform convergence $\{x^n\}$ for $x\in[0,1]$.
15. To prove the Bolzano-Weierstrass theorem in spaces R^3, R^2.
16. Show that from the uniform convergence of the sequence of functions $\{X_n(x)\}$ its convergence over distance in space follows C_{L_2}. Is the converse statement true?
17. Prove that from the convergence of the sequence over the space distance l_2 the coordinate convergence follows. Is the reverse statement true?

MAPPING OF METRIC SPACES

Limit and continuity. Criterion of continuity.

Let us have two metric spaces (X,ρ_x), (Y,ρ_y)) and the set $E\subset(X,\rho_x)$.

Definition. If to each element $x\in E\subset\left(X,\rho_x\right)$ a completely specific element in space (Y,ρ_y) is matched, then we say that the mapping of the set E into the space is given (Y,ρ_y) and mark: $y=f(x)$, or $y=Ax$.

The concept of mapping is a generalization of the concept of function. It can be a numerical function of one variable, many variables, a functional. In functional analysis, the mapping, which to each element of arbitrary nature x an element of arbitrary nature y is matched, is also called an operator.

Examples.

1) $R^1\to R^1:\ y=\sin x$;

2) $R^n\to R^1:\ y=f(x_1,x_2,...,x_n)$;

3) $C_{[a,b]}\to R^1:\ y=\int\limits_a^b x(t)dt$ – functional;

4) $R^n\to R^m:\ \begin{cases} y_1=a_{11}x_1+a_{12}x_2+...+a_{1n}x_n+b_1; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_m=a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n+b_m; \end{cases}$

5) $C_{[a,b]}\to C_{[a,b]}:\ x(t)\to x'(t)$;

6) Let us consider the Bernstein polynomial: $B_n(f,x)=\sum\limits_{k=0}^{n}C_n^k f(\frac{k}{n})x^k(1-x)^{n-k}$. This is the mapping to every continuous function $f(x)$ matches the Bernstein polynomial. Function $f(x)$ is set on the segment [0,1]. The peculiarity of the Bernstein polynomial is that it can approximate any continuous function as accurately as desired $f(x)$ on [0,1].

Due to the power of the polynomial n, it is possible to achieve that the difference between the function $f(x)$ and the Bernstein polynomial will be arbitrarily small.

Let x_0 – the limit point of the set E, and it does not necessarily belong to the set E, which is included in the space (X,ρ_x).

Definition (by Cauchy). Element A, which belongs to space (Y,ρ_y) is called the *mapping limit* $y = f(x)$ at $x \to x_0$, if for $\forall \varepsilon > 0\ \exists \delta > 0$, so that, as soon as $\rho(x, x_0) < \delta$, then $\rho(f(x), A) < \varepsilon$.

Definition (by Heine). Element A is called the mapping limit $y = f(x)$ при $x \to x_0$, if from the convergence of any sequence $\{x_n\} \to x_0$ the convergence of the corresponding sequence of mapping values follows $f(x_n)$ to A. (Here, convergence is understood as convergence by distance).

Now let the mapping be defined at a point x_0.

Definition. The mapping $y = f(x)$ is called *continuous at a point* x_0, if for $\forall \varepsilon > 0\ \exists \delta > 0$, that as soon as $\rho(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \varepsilon$.

Definition. The mapping $y = f(x)$ is called *continuous at a point x_0,* if the convergence of any sequence $\{x_n\} \to x_0$ implies convergence $\{f(x_n)\} \to f(x_0)$.

If a mapping is mutually unambiguous, then there is an inverse mapping for it.

Definition. A mutually unique and continuous mapping is called a *homeomorphism*.

Definition. A homeomorphic mapping is called an *isometry*, or *isomorphism*, if it preserves the distance between images.

Theorem (the criterion of continuity). In order for the mapping $y = f(x)$ to be continuous, it is necessary and sufficient that the prototype $f^{-1}(G)$ was open for every open set G included in Y.

Proof. Necessity. We will take into account what the mapping $y = f(x)$ – is continuous and the set $G \subset Y$ – is open. It is necessary to prove that $f^{-1}(G)$ – is open, that is, all its points are internal. Suppose that $x_0 \in f^{-1}(G)$, $y_0 = f(x_0) \in G$. Let us show that the point x_0 belongs to the set $f^{-1}(G)$ together with his neighborhood. Since y_0 is an interior point of G, then exists $\varepsilon > 0$, that sphere $B(y_0,\varepsilon) \subset G$. Due to the continuity of the mapping according to the given ε we will find $\delta>0$, that as soon as x falls in to the ball $B(x_0,\delta)$, then y falls in to the ball $B(y_0,\varepsilon)$. It is clear that the sphere $B(x_0,\delta)$ belongs to the set of preimages $f^{-1}(G)$, it means that $x_0 \in f^{-1}(G)$ together with some of its neighborhood, which means that $f^{-1}(G)$ is an open set (due to the arbitrariness of the chosen point x_0).

Sufficiency. Given $f^{-1}(G)$ – an open sphere, for any open set $G \in Y$. Lets prove that the mapping is continuous. Let x_0 – arbitrary point of the set $f^{-1}(G)$ and $y_0 = f(x_0)$. For $\forall \varepsilon > 0$ consider the sphere $B(y_0,\varepsilon) \subset G$. By condition, the image of this sphere: $f^{-1}(B(y_0,\varepsilon))-$ is an open set, i.e., all its points are interior, and therefore also the point x_0, which means that there exists such $\delta > 0$, that sphere $B(x_0,\delta)$ get into the sphere $f^{-1}(B(y_0,\varepsilon))$, and this means the continuity of this mapping at the point x_0 according to Cauchy.

The theorem is proved.

Connectivity and its preservation during continuous mapping.

Definition. A set E is called *connected* if it cannot be represented as the union of two disjoint open sets.

Theorem. The image of a connected set under a continuous mapping is a connected set.

Proof. Given $G \subset (X,\rho)$ – is connected set. It is necessary to prove that $f(G) \subset Y$ – is connected. Let $f(G) = E$ – is not connected, and it is presented as the union of two open sets that do not intersect. Let's denote them by E_1 and E_2. But according to the criterion of continuity, the preimage of the set E_1 is an open set, and the preimage of E_2 is also open. Let's label them G_1 and G_2, they will not intersect (due to the continuity of the mapping). That is, we presented the set of preimages in the form of the union of two disjoint open sets, which contradicts the fact that G is connected. So the image $f(G)$ – is a connected set.

The theorem is proved.

***Control questions*.**

1. Give the definition of mapping a set $E \subset (X, \rho_x)$ into space (Y, ρ_y).
2. Give the definition of continuity of mappings and homeomorphism.
3. Formulate a criterion of continuity.
4. Formulate a theorem on the connectivity of an image with a continuous mapping.
5. Give the definition of uniform continuity of the mapping.

Exercises.

1. Give examples of reflections from space R^1 to R^1; from R^2 to R^1; from $C_{[a,b]}$ to R^1; from $C_{[a,b]}$ to $C_{[a,b]}$.
2. Given the Fredholm operator $Ax(t) = \int_a^b K(t,s)x(s)ds$, moreover, all functions appearing here are continuous on closed sets. Prove the continuity of this mapping.
3. Will the representation A of space be continuous $C_{[0,1]}$ in itself, which acts according to the formula $Ax(t) = a(t)x(t)$, де $a(t)$ – given continuous function.
4. The function is given $f(x) = \sqrt{x+4}$. Prove in language "δ–ε" continuity of a function at a point x=5.

The solution.

1. Examples of such mappings can be mappings: $y=\sin x$, $z=x^2+y^2$, $y=\int_a^b x(t)dt$, $y(x)=x_0(t)x(t)$, де $x_0(t)$ – is a fixed function, and $x(t)$ – is an arbitrary.

2. We will prove the continuity of the mapping according to Heine, that is, in the language of sequences. Let us have a sequence of continuous functions $x_n(s)$, which are heading to x(s) in the sense that $\max_S |x(s)-x_n(s)| \to 0$, at $n\to\infty$. Then

$$|Ax(t)-Ax_n(t)| = \left|\int_a^b K(t,s)(x(s)-x_n(s))ds\right| \le \int_a^b M\cdot|x(s)-x_n(s)|ds =$$

$$= M\max_S |x(s)-x_n(s)|(b-a)\to 0, \text{ at } n\to\infty,\ (M=\max|K(t,s)|).$$

Continuity of mapping is proved.

3. This mapping will be continuous. This can be verified in exactly the same way as it was done in the previous example.

4. At the point x=5 the function is defined: $f(5)=3$. Let's set $\varepsilon>0$. Let's add up the difference $f(x)-f(5)=\sqrt{x+4}-3$, let's evaluate it by the module. At $\delta>0$ for values of x that satisfy the inequalities $|x-5|<\delta$, the inequality will also be satisfied:

$$\left|\sqrt{x+4}-3\right| = \frac{|x-5|}{\left|\sqrt{x+4}+3\right|} < \frac{|x-5|}{3} < \frac{\delta}{3}.$$

If now put $\delta=3\varepsilon$, then at values of x for which $|x-5|<3\varepsilon$, the inequality will hold $\left|\sqrt{x+4}-3\right|<\varepsilon$.

Continuity of the function at $x=5$ is proved.

Problems.

1. Give examples of reflections: R^1 to R^1; from R^n to R^1; from R^n to R^n; from $C_{[a,b]}$ to R^1; from $C_{[a,b]}$ to $C_{[a,b]}$; from R^1 to R^n.

2. Prove by definition in language “δ−ε” continuity of function $y=x^2$ at the point x_0.
3. Prove that the functions $y=x$ and $y=\sin x$ are uniformly continuous on the entire number axis.
4. To prove that the function $y=\frac{1}{x}$ is continuous on the interval $[0,1]$, but is not uniformly continuous on this interval.
5. Give an example of a continuous mapping that is not a homeomorphism.
6. 6. Prove the continuity of a numerical function $\rho(x,y)$ in any metric space.
7. Prove the continuity of a vector function $y=(\varphi_1(t),\varphi_2(t),...,\varphi_n(t))$ of the numerical argument *t*, if coordinate functions $\varphi_n(t)$ continuous.
8. Prove the continuity of the mapping:

$-\ y = x(t_0),\ x(t)\in C_{[a,b]},\ t_0$ – segment fixed point $[a,b]$.

$-\ y=\int_a^b x(t)dt,\ x(t)\in C_{[a,b]}$.

$-\ y=a_1x_1+a_2x_2+...+a_nx$, де $x=(x_1,x_2,...,x_n)$ – arbitrary vector, $a=(a_1,a_2,...,a_n)$ – fixed vector.

$$-\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + b_1 \\ y_2 = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + b_2 \\ .... \\ y_n = a_{1n}x_1 + a_{2n}x_2 + ... + a_{nn}x_n + b_n \end{cases};$$

Here is the matrix of coefficients (a_{ij}) and a column of free members (b_s), $(i=1,2,...,n)$ are set, $(x_1,x_2,...,x_n)$ is an arbitrary vector variable x.

COMPLETENESS OF METRIC SPACES

In mathematical analysis, the Cauchy criterion of convergence of a numerical sequence plays a fundamental role (in order for the sequence to converge, it is necessary and sufficient for it to be fundamental). This criterion characterizes the completeness of the metric space R, or the number line (there are no holes on the number line). But this criterion ceases to be fair in an arbitrary space.

<u>Example.</u> Indeed let us have an interval (0,1). We set the distance as follows: $\rho(x,y)=|x-y|$. Consider the sequence: $\left\{\frac{1}{n}\right\}$, $n \in N$. It is obvious that at $n \to \infty$, the limit of our sequence will be zero, but zero does not belong to our space.

Consider another example: a set of rational numbers and a sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$, at $n \to \infty$ the limit of this sequence will be the number *e*, which is irrational (it does not belong to the set of rational numbers).

<u>Definition.</u> Metric space (X,ρ) is called *complete* if any fundamental sequence in this space has a limit.

<u>Example.</u> Let's explore the space (X,ρ) on completeness, $X \subset N$, $\rho(x,y)=|x-y|$. Consider an arbitrary fundamental sequence composed of elements of such a space. Such a sequence here can be the following sequence: a finite number of the first *n* members of the sequence will be arbitrary elements, and starting from the *(n+1),* some one of the elements will be repeated (there can be no other cases). The limit of such a sequence will obviously be a repeating element, it will belong to the space *N*, so such a space is complete by definition.

An important role in mathematical analysis is played by Cantor's theorem about a system of nested and compressing segments, the lengths of which go to zero, which states that such a system has one common point. This fact expresses the fact that the number line is complete. A similar role is played by the theorem on nested spheres.

Theorem 1. Any system of nested closed balls whose radii goes to zero has a nonempty intersection in the complete metric space.

Proof. Given:

1) Sequence of nested balls $B_1(x_1,r_1)\supset B_2(x_2,r_2)\supset\ldots\supset B_n(x_n,r_n)\supset\ldots,$

2) Balls B_i – are closed,

3) $\lim\limits_{n\to\infty} r_n = 0$,

4) The space is complete.

Prove that balls do not have an empty cross section. Consider the sequence of ball centers $x_1, x_2,\ldots, x_n$. This sequence will be fundamental. Indeed, at $n\to\infty$ and $m>n$, $\rho(x_n,x_m)<r_n \to 0,$ which means fundamentality. Due to the completeness of space, this sequence has a limit $\lim\limits_{n\to\infty} x_n = x$. Let's show that x belongs to all balls. Indeed, x is the limit point for each ball, because in any of its neighborhoods there are an infinite number of points of each of the balls. Due to the closedness of the balls, x will belong to each ball, which means it belongs to the intersection.

Theorem 2. A closed subspace of the complete space is complete.

Proof. Given (X,ρ) – complete space and $(Y,\rho)\subset(X,\rho)$ – closed space. Prove, that (Y,ρ) – complete. Consider an arbitrary fundamental sequence $\{y_n\}\subset(Y, \rho)\subset(X, \rho)$. Space (X,ρ) – is a complete space, so this sequence has a limit in it: $\lim\limits_{n\to\infty} y_n = y \in (X,\rho)$. But y is the limit point of the set (Y, ρ), and (Y, ρ) – is closed, which means that it contains all its limit points, $y \in (Y,\rho)$.

The theorem is proved.

Completeness of some important metric spaces.

Theorem 1. The space R^n – is complete.

Proof. Consider an arbitrary fundamental sequence in this space $\{(x_1^{(m)}, x_2^{(m)}, \dots, x_n^{(m)})\}$. So, $\forall\ \varepsilon > 0\ \exists\ N$, that as soon as $m,k > N$, than $\rho(x^{(m)}, x^{(k)}) < \varepsilon \Rightarrow \sqrt{(x_1^{(m)} - x_1^{(k)})^2 + \dots + (x_n^{(m)} - x_n^{(k)})^2} < \varepsilon \Rightarrow$

$| x_1^{(m)} - x_1^{(k)} | < \varepsilon$,…, , and these are the fundamental conditions of the sequences of the corresponding coordinates in space R^1, but space R^1 – is complete, therefore there will be limit: $\lim\limits_{n\to\infty} x_1^{(m)} = x_1^*$,…, $\lim\limits_{n\to\infty} x_n^{(m)} = x_n^*$.

Let's construct the element of space R^n $(x_1^*, x_2^*, \dots, x_n^*)$, which will be the limit of our fundamental sequence (since in this space coordinate convergence is equivalent to distance convergence).

The theorem is proved.

Theorem 2. The space $C_{[a,b]}$ – is complete.

Proof. Let $\{x_n(t)\}$ – an arbitrary fundamental sequence. For $\forall \varepsilon > 0\ \exists N$, $n, m > N$, $\rho(x_n(t), x_m(t)) = \max\limits_t |x_n(t) - x_m(t)| < \varepsilon$. We justify the existence of the limit of this sequence. Let's fix it $t_0 \in [a,b]$, we will get a numerical sequence $\{x_n(t_0)\}$, which will be fundamental, and therefore in this space it will have a limit $x(t_0)$. Since t_0 we chose arbitrarily, then there will be a limit $\forall t \in [a,b]$. Let's denote it through $x(t)$.

By moving in inequality $\max\limits_t |x_n(t) - x_m(t)| < \varepsilon$ to the limit when $m \to \infty$, we will get $\max\limits_t |x_m(t) - x(t)| < \varepsilon$. The last inequality means that $x_m(t)$ goes to the limit function uniformly. Then according to the well-known theorem of mathematical analysis $x(t)$ continuous.

The theorem is proved.

Theorem 3. The space l_2 – is complete.

Proof. Consider an arbitrary fundamental sequence in this space $\{(x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)}, \dots)\}$. And a series $\sum\limits_{k=1}^{\infty} x_k^2 < \infty$ – convergent. Let's write down the fundamentality condition: $\forall\varepsilon > 0\ \exists N,\ n,m > N$:

$$\rho(x_n, x_m) = \sqrt{\sum_{k=1}^{\infty} (x_k^{(n)} - x_k^{(m)})^2} < \varepsilon \Rightarrow$$

$$|x_1^{(n)} - x_1^{(m)}| < \varepsilon,$$

....

$$|x_k^{(n)} - x_k^{(m)}| < \varepsilon,$$

....

The last inequalities mean that the numerical sequences of the coordinates are fundamental. But the space R^1 is complete, so there are limits:

$$\lim_{n\to\infty} x_1^{(n)} = x_1,$$

....

$$\lim_{n\to\infty} x_k^{(n)} = x_k,$$

....

Let's construct such a vector $x = (x_1, x_2, ...x_k, ...)$. It will be the limit of the original fundamental sequence. For this it is enough to show that

1) sequence $\{x_n\} \to x$ according to the metric of this space;

2) $x \in l_2$, that is, the series $\sum_{k=1}^{\infty} x_k^{\ 2}$ – is convergent.

Let's briefly prove the first fact.

Since sequence is fundamental, inequality will be fair: $\sum_{k=1}^{\infty}(x_k^{(n)} - x_k^{(m)})^2 < \varepsilon^2$, for $\forall\varepsilon > 0$ and such N, that $n,m > N$. Then for $\forall M \in N$ we will have: $\sum_{k=1}^{M}(x_k^{(n)} - x_k^{(m)})^2 < \varepsilon^2$. Let's make the limit transition at $m \to \infty$, and we will have: $\sum_{k=1}^{M}(x_k^{(n)} - x_k)^2 < \varepsilon^2$. Since M is arbitrary, we have: $\sum_{k=1}^{\infty}(x_k^{(n)} - x_k)^2 < \varepsilon^2$, and this means that the selected sequence goes to the limit vector: $x=(x_1,x_2,...x_k,...)$.

Let's prove the second fact, namely what $x\in l_2$, that is, the series $\sum_{k=1}^{\infty} x_k{}^2<\infty$ – is convergent. Let's use the inequality $(a-b)^2\leq 2(a^2+b^2)$. Having set $a=x_k^{(n)}$, $b=x_k^{(n)}-x_k$, and, substituting them into an inequality, we get: $x_k^2\leq 2(x_k^{(n)2}+(x_k^{(n)}-x_k)^2)$. Series of which members are $x_k^{(n)2}$ and $(x_k^{(n)}-x_k)^2$ are convergent, hence it follows that the series whose members are x_k^2 will also be convergent, which means that the limit element $x\in l_2$. Therefore, l_2– is complete.

The theorem is proved.

Definition. Metric space R^* is called a *completion* of the space R if three conditions are satisfied:

1. Space $R^*\supset R$,
2. R dense everywhere in space R^*,
3. R^* – is complete.

Theorem. Any incomplete metric space R can be completed in only one way.

Control questions.

:

1. Give the definition of a complete metric space.
2. Formulate a theorem about nested spheres.
3. Prove that a closed subspace of a complete space is complete.
4. Prove the completeness of spaces: R^n, l_2, $C_{[a,b]}$.
5. Formulate the theorem on replenishment of space.

Exercises.

1. Given a set of natural numbers $\rho(x,y)=|x-y|$. Prove that this space is complete.

2. Given a set of natural numbers N, $\rho(m,n)=\dfrac{(m-n)}{mn}$. Prove that this space is not complete.
3. Given a set of irrational numbers I_{pp}, $\rho(x,y)=|x-y|$. Explore this space to the fullest.
4. Explore the space in its entirety R_1^n.

The solution.

1. Consider an arbitrary fundamental sequence of natural numbers in this space. It will necessarily have the following structure: a finite number of its first members will be arbitrary natural numbers, and starting from a certain number, one of the natural numbers will be repeated, otherwise the sequence will not be fundamental. Then it is clear that this is a natural number that repeats itself and will be the limit of the sequence.

2. This space is not complete. To prove this, it is enough to indicate at least one fundamental sequence, which will have no limit in this space. Such a sequence will be the sequence of natural numbers *1,2,3...,n,...* Indeed, this sequence according to this metric will be fundamental, because the expression $\dfrac{|m-n|}{mn}$ can be made arbitrarily small for sufficiently large *m* and *n*. However, this sequence will have no limit. Because if such a limit existed and were equal to the natural number *a*, then we would have what $\rho(m,a)=\dfrac{|m-a|}{ma}\to 0$. In fact, for example when $m>a$ we have $\dfrac{|m-n|}{mn}=\dfrac{|m-a|}{ma}=\dfrac{1}{a}-\dfrac{1}{m}=\dfrac{1}{a}\neq 0$ at $m\to\infty$.

3. This space is not complete because, for example, the fundamental sequence $\dfrac{\sqrt{2}}{n}$ there is no limit to irrational numbers in this space. (The sequence goes to zero, which is not an irrational number).

4. Consider an arbitrary fundamental sequence in this space $\{(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)})\}$. It means that for any $\varepsilon>0$ there exists a number N such that at $n, k \geq N$, the inequality is fulfilled $f(x^n, x^k) = \sum_{i=1}^{m} \left| x_i^{(n)} - x_i^{(k)} \right| < \varepsilon \Rightarrow \left| x_i^{(n)} - x_i^{(k)} \right| < \varepsilon$ i=1,2,...,m. This means that the numerical sequences of the coordinates are fundamental. But the space R^1 is complete, so these numerical sequences will have limits: $\lim_{n\to\infty} x_i^{(n)} = x_i$, i=1,2,...,m. Let's add a vector $x=(x_1, x_2, ..., x_m)$. It will be the limit of the original sequence, because the convergence is by distance in space R_1^n equivalent in coordinate convergence.

Problems.

1. Prove the completeness of spaces R_1^n, R_0^n.
2. Prove that the space of isolated points is complete.
3. Investigate thoroughly C_L, C_{L_2}.
4. Given $x \in (0,1)$, $\rho(x,y)=|x-y|$. Explore this space to the fullest.
5. Given a set Q of rational numbers, $\rho(x,y)=|x-y|$. Explore this space to the fullest.
6. Given a set I_{pp} numerical axis, $\rho(x,y)=|x-y|$. Explore this space to the fullest.
7. Explore the segment in its entirety $[a,b]$ if $\rho(x,y)=|x-y|$.
4. Prove that the subspace of functions *f(x),* which belong to space $C_{[a,b]}$ and which satisfy the condition $A \leq f(x) \leq B$ is complete.
8. Given a set of continuous functions that have continuous derivatives, and $\rho(f,g) = \sup|f(x) - g(x)| + \sup|f'(x) - g'(x)|$. Prove that this space is complete.

9. Given a metric space (X,ρ): $x=(x_1,x_2,\ldots,x_n,\ldots)$, sum $\sum_{k=1}^{\infty}|x_k|<\infty$,

$\rho(x,y)=\sum_{k=1}^{\infty}|x_k-y_k|$. Prove that this space is complete.

BANACH'S THEOREM AND ITS APPLICATIONS

There are many theorems related to the existence of a solution to equations ($F(x)=0$,differential equations, integral, SLAE), which can be proved from a single point of view: by proving the existence of a fixed point of some reflection.

Definition. The point x is called a *fixed point* of the metric space mapping (X,ρ) in itself, if equality is fair $Ax=x$.

Definition. The mapping $y = Ax$ of a metric space (X,ρ) to itself is called *compressive*, if such a number exists α: $0\le\alpha<1$ that the inequality is fulfilled $\rho(Ax,Ay)\le\alpha\cdot\rho(x,y)$. That is, the distance between the images does not exceed the distance between the prototypes.

Theorem. All compressive mapping is continuous.

Proof. We will carry out the proof in the language of sequences, that is, according to Heine. Let's $\{x_n\}$ – an arbitrary sequence that matches an element x_0. That is, $\rho(x_n,x_0)\to 0$, at $n\to\infty$. Let's write down the compression condition: $0\le\alpha<1$, $\rho(Ax_n,Ax_0)<\alpha\rho(x_n,x_0)$. At $n\to\infty$, $\rho(x_n,x_0)\to 0$, therefore $\alpha\rho(x_n,x_0)\to 0$, whence the expression on the left also goes to zero, which means that the sequence $\{Ax_n\}\to\{Ax_0\}$. Whence follows the continuity of the mapping.

Terema is proven.

Theorem (Banach). Any compression mapping that translates completed space (X,ρ) in itself, has one and only one fixed point x in this space, or what is the same as equation $Ax=x$ has a single root x.

Proof. Given: (X,ρ) – the complete space; $Ax \subset (X,\rho)$ – self-mapping;

$\rho(Ax, Ay) \leq \alpha \cdot \rho(x, y)$, $0 \leq \alpha < 1$ – compression mapping.

It is necessary to prove that there is an element $x \in (X,\rho)$,that $Ax=x$.

Let's x_0 – arbitrary point in space (X,ρ). Let's build the sequence in this way:

$x_1 = Ax_0$,

$x_2 = Ax_1$,

....

$x_n = Ax_{n-1}$.

Received in space (X,ρ) some sequence $\{x_n\}$. We will show that such a sequence is fundamental. Without reducing the generality, we will assume that $n<m$. Let's evaluate:

$$\rho(x_n, x_m) = \rho(Ax_{n-1}, Ax_{m-1}) \leq \alpha\rho(x_{n-1}, x_{m-1}) = \alpha\rho(Ax_{n-2}, Ax_{m-2}) \leq \alpha^2\rho(x_{n-2}, x_{m-2}) \leq ..$$
$$... \leq \alpha^m \rho(x_{n-m}, x_0),$$

$\alpha^m \to 0$, when $m \to \infty$.

The behavior of the second multiplier remains unknown. Since the space is metric, we can use the triangle inequality:

$$\rho(x_{n-m}, x_0) \leq \rho(x_0, x_1) + \rho(x_1, x_{n-m}) \leq \rho(x_0, x_1) + \rho(x_1, x_2) + ... + \rho(x_{n-m-1}, x_{n-m}).$$

Let's evaluate each term starting from the second of this inequality:

$\rho(x_1, x_2) = \rho(Ax_0, Ax_1) < \alpha\rho(x_0, x_1)$,

$\rho(x_2, x_3) = \rho(Ax_1, Ax_2) < \alpha\rho(x_1, x_2) < \alpha^2\rho(x_0, x_1)$,

Similarly, we will have:

$\rho(x_3, x_4) < \alpha^3\rho(x_0, x_1)$,

…………………………

$\rho(x_{n-m-1}, x_{n-m}) < \alpha^{n-m-1} \cdot \rho(x_0, x_1)$.

Next is obvious:

$$\rho(x_n,x_m)\le\alpha^m\rho(x_0,x_1)(1+\alpha+\alpha^2+...+\alpha^{n-m-1})\le\frac{\alpha^m}{1-\alpha}\cdot\rho(x_0,x_1).$$

As $\alpha^m\to 0$, when $m\to\infty$, and $\frac{\rho(x_0,x_1)}{1-\alpha}$ is some fixed constant, then the entire expression on the right side of $m\to\infty$ also goes to zero. This would mean that our sequence is fundamental. Since our space is complete, the fundamental sequence will have a limit in this space: $\lim\limits_{n\to\infty}x_n=x$.

Let us show that x will be a fixed mapping point, namely $Ax=x$. We have: $\lim\limits_{n\to\infty}x_n=x$, a $x_n=Ax_{n-1}$. The mapping A is continuous, so if $x_n\to x$ at $n\to\infty$, than Ax_{n-1} will be heading to Ax, that is, we will have that $Ax=x$ at $n\to\infty$, which means that x – fixed mapping point.

Let us show that this point is unique.

Let there exist y such that $Ay=y$, then we will have that $\rho(x,y)=\rho(Ax,Ay)\le\alpha\rho(x,y)$. So we got that: $\rho(x,y)\le\alpha\rho(x,y)$, this is possible only when $\rho(x,y)$=0. Since the space is metric, then from the first metric axiom follows that $x\equiv y$.

The theorem is proved.

Remark 1. In the process of proving Banach's theorem, we not only proved the existence of a fixed point, but also indicated the method of its approximate finding. This method is called the method of successive approximations. The final result does not depend on the choice of the zero approximation x_0. This fact is of considerable interest from a computational point of view, because we can take each of the following approximations as x_0, which will prevent the accumulation of errors that will depend on the initial approximation.

Remark 2. In practice, calculations are stopped at some step. Then the question arises, how to estimate the error between the exact result and the approximate one. Let's use the inequality obtained in the process of proving the theorem:

$\rho(x_n,x_m)=\dfrac{\alpha^m}{1-\alpha}\rho(x_0,x_1)$, at $m\to\infty$ we have: $\rho(x_n,x)=\dfrac{\alpha^m}{1-\alpha}\rho(x_0,x_1)$. This estimate shows that successive approximations matches to the exact solution at the rate of geometric progression.

The simplest applications of Banach's theorem.

Let there be an equation $x=f(x)$, where $f(x)$ – is a continuous function of one variable defined on segment $[a,b]$ and its values do not go beyond the segment $[a,b]$ (we will have a mapping in ourselves). In addition $[a,b]$ is a complete space as a closed subspace of the complete space. Let the compression condition hold, i.e: $|f(x_2)-f(x_1)|\le\alpha\cdot|x_2-x_1|$, $0\le\alpha<1$. Therefore, there will be a single root of the equation $x=f(x)$, which we can find by the method of successive approximations. Note that the compression condition, which is difficult to verify in practice, will hold if $f(x)$ has a derivative such that: $|f'(x)|<M<1$. Indeed, using Lagrange's theorem, we have: $f(x_2)-f(x_1)=f'(c)\cdot(x_2-x_1)$, $c\in[x_1,x_2]$. Then, taking the left and right parts modulo, we get:

$|f(x_2)-f(x_1)|=|f'(c)|\cdot|x_2-x_1|\Rightarrow|f'(c)|<1$. Consider the more complicated case of equation: $F(x)=0$. The original equation is equivalent to such an equation: $-\lambda F(x)=0$. Let's add to both parts x: $x-\lambda F(x)=x$, obtained the equation $f(x)=x$ from the previous case. We will demand that compression is performed, there is a reflection in itself, and that the space is full (space R^1 is full). We demand that: $\left|\left(x-\lambda F(x)\right)'\right|<1$, тобто $|1-\lambda F'(x)|<1$, $-1<1-\lambda F'(x)<1$, $\Rightarrow$

$-2<-\lambda F'(x)<0\Rightarrow 0<\lambda F'(x)<2\ \Rightarrow\lambda<\dfrac{2}{\max\limits_x F'(x)}$.

Consider a system of n linear equations with n unknowns:

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + b_1, \\ .... \\ y_n = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + b_n. \end{cases}$$

Let's set the problem of solving a system of equations in which instead y_i will stand instead x_i, $i = \overline{1, n}$.

Firstly, R^n – the space is full. Secondly, we have a reflection in ourselves ($R^n \to R^n$). Third, let's clarify the fulfillment of the compression condition, which will depend on the space metric R^n. Let us have space R_0^n, then: $\rho(x'', x') = \max_i |x_i'' - x_i'|$, accordingly we will have: $\rho(y'', y') = \max_i |y_i'' - y_i'|$.

We have:

$$\rho(y'', y') = \max_i |y_i'' - y_i'| = \max_i \left| \sum_{j=1}^n a_{ij}(x_j'' - x_j') \right| \le \max_i \sum_{j=1}^n |a_{ij}| \cdot |x_j'' - x_j'| \le$$

$$\le \max_i \sum_{j=1}^n |a_{ij}| \cdot \max_j |x_j'' - x_j'| = \max_i \sum_{j=1}^n |a_{ij}| \cdot \rho(x'', x').$$

Therefore, the compression condition will be as follows: $\alpha = \max_i \sum_{j=1}^n |a_{ij}| < 1$. Similarly, compression conditions for spaces are proved R_1^n: $\alpha = \max_j \sum_{i=1}^n |a_{ij}| < 1$, R^n: $\alpha = \sqrt{\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2} < 1$.

When one of the three compression conditions is met, the solution of the system can be found using Banach's theorem. The zero approximation can be taken as a column of free members: $x^0 = (b_1, b_2, ..., b_n)$. By inserting it into the system, we will get a first approximation. Continuing this process, we will obtain a sequence of approximations that leads to the exact solution. In order to obtain an approximation with a certain accuracy, use the estimate: $\rho(x, x_n) < \dfrac{\alpha^m}{1-\alpha} \cdot \rho(x_0, x_1)$.

Application of Banach's theorem to prove the existence and unity of the solution of the Cauchy problem of the differential equation $y' = f(x, y)$.

Let us have a differential equation $y' = f(x, y)$. It is known that it has many solutions that depend on some parameter *c*. A solution that satisfies the initial condition $y(x_0) = y_0$ is called partial. Finding a partial solution to such a differential equation is called the Cauchy problem. It is difficult to solve it, sometimes it is absolutely impossible. Then they solve approximately. But at the same time, one must be sure that there is only one solution. This is proved using Banach's theorem.

Theorem. Let us have the equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$, where is a function $f(x, y)$ – defined and continuous in some region *D* containing a point (x_0, y_0), and satisfies the Lipshitz condition on the variable in this domain *y*: $\forall y_1, y_2 \in D$ $|f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1|$. Then there is such a segment $|x - x_0| \le \alpha$ on which there is a single function $y = y(x)$, which is a solution of the original equation and satisfies the initial condition.

Proof. First, we will prove the equivalence of the initial differential equation with the initial condition and the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt .$$

Let $y'(x) = f(x, y)$, $y(x_0) = y_0$, (1).

Let's integrate this equation in the range from x_0 to *x*, considering $y(x)$ it continuous.

$$\int_{x_0}^{x} y'(t)dt = \int_{x_0}^{x} f(t, y(t))dt,$$

$$y(t)\Big|_{x_0}^{x} = \int_{x_0}^{x} f(t, y(t))dt\,;$$

$$y(x) = \underbrace{y(x_0)}_{y_0} + \int_{x_0}^{x} f(t, y(t))dt \quad (2).$$

Now let *y(x)* be the solution of equation (2). Since $f(x, y)$ – is continuous and *y(t)* is continuous, then their superposition is also continuous. Then we can take the derivative of the integral with an upper variable limit, and we get: $y'(x) = f(x, y(x))$, $y(x_0) = y_0$.

We will prove the existence of a solution to the integral equation. Since $f(x, y)$ continuous in D, then it will be continuous and in some case closed $\overline{D}$, $\overline{D} \subset D$. Then according to the Weierstrass theorem $f(x, y)$ will be limited to some number *k* in $\overline{D}$: $|f(x, y)| < k$. Let's pick up α, which appears in the condition of the theorem, so that:

1) $|x - x_0| < \alpha$, $|y - y_0| \le k\alpha$, then $(x, y) \in \overline{D}$

2) $M\alpha < 1$, M – constant from the Lipshitz condition.

Consider space C^*, whose elements are continuous functions $\varphi(x)$ on the segment $|x - x_0| \le \alpha$, which satisfy the condition $|\varphi(x) - y_0| \le k\alpha$.

Let's consider in this space C^* operator $\psi = A\varphi$, $\psi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt$.

We will show that Banach's theorem can be applied to this operator, which ensures the existence of a fixed point of this operator, or that the same is the solution of the integral equation (2), and therefore its equivalent equation (1). Let's check the fulfillment of the conditions of Banach's theorem.

1. Space C^* – complete, as a closed subspace of complete space $C_{[a,b]}$.

2. We have a reflection in ourselves, really: $\psi(x)$ – continuous, as

a superposition of continuous functions. In addition:

$$|\psi(x)-y_0|=\left|\int_{x_0}^{x} f(t,y(t))dt\right|\le\left|\int_{x_0}^{x}|f(t,y(t))|dt\right|<\left|\int_{x_0}^{x} kdt\right|=k\,|x-x_0|\le k\alpha\,.$$

3. Let's check the compression condition:

$$|\psi_2(x)-\psi_1(x)|=\left|\int_{x_0}^{x} f(t,\varphi_2(t))dt-\int_{x_0}^{x} f(t,\varphi_1(t))dt\right|\le\left|\int_{x_0}^{x}|\varphi_2(t)-\varphi_1(t)|Mdt\right|\le$$

$$\left|\int_{x_0}^{x}\max_t|\varphi_2(t)-\varphi_1(t)|Mdt\right|=M\rho(\varphi_1,\varphi_2)\left|\int_{x_0}^{x} dt\right|=M\rho(\varphi_1,\varphi_2)|x-x_0|\le M\alpha\rho(\varphi_1,\varphi_2).$$

The inequality holds for all *x*, then for $\max_x$ also: $\rho(\psi_1,\psi_2)\le M\alpha\rho(\varphi_1,\varphi_2)$. In the assumption that $M\alpha<1$, the compression condition is satisfied. All three conditions of Banach's theorem are fulfilled, so there is a fixed point of the operator $\psi=A\varphi$.

Remark. The condition of the theorem contains the Lipshitz condition, it is difficult to verify in practice, so it is replaced by a simpler one, namely, it requires that $f'_y(x,y)$ was continuous, then according to the Weierstrass theorem $|f'_y(x,y)|\le M$;

$|f(x,y_2)-f(x,y_1)|=|f'_y(x,c)||y_2-y_1|$,

$|f(x,y_2)-f(x,y_1)|\le M|y_2-y_1|$.

Remark. In this method, the approximate solution approaches the exact solution at the rate of geometric progression. Here is a fair assessment:

$|y(x)-y_n(x)|\le\dfrac{\alpha^n}{1-\alpha}|y_1(x)-y_0(x)|$, for, any *x*. In addition, errors do not accumulate in the process of calculations.

Application of Banach's theorem to prove the existence of the Cauchy problem of a normal system of differential equations.

The system of the species:

$$\begin{cases} y_1'(x) = f_1(x \ , y_1, y_2 .....y_n), \\ .... \\ y_n'(x) = f_n(x \ , y_1, y_2 .....y_n). \end{cases} \qquad (1)$$

is called a normal system of differential equations. Here, each equation is solved with respect to the derivative. We have n equations with n unknown functions $y_1, y_2, ..., y_n$. Let us have the initial conditions:

$$y_1(x_0) = y_{01},$$

$$y_2(x_0) = y_{02}, \qquad (1^*)$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_n(x_0) = y_{0n},$$

System functions $y_1(x), ..., y_n(x)$ changes in x on some interval are called a solution of system (1) if, when substituting into this system, we obtain identities.

Finding solutions $y_1, y_2, ..., y_n$, which satisfy the conditions (1*) is called the Cauchy problem. Solving this problem is more difficult than the previous one.

Theorem. Let us have a normal system of differential equations of the form (1) with conditions (1*), where functions $f_i \ (i = \overline{1,n})$ defined and continuous in space R^{n+1}, which contains a dot $(x_0, y_1, ..., y_n)$, and f_i satisfy the Lipshitz condition in the form

$$\left| f_i(x, y_1^{(2)}, y_2^{(2)}, ..., y_n^{(2)}) - f_i(x, y_1^{(1)}, y_2^{(1)}, ..., y_n^{(1)}) \right| \le M \max_i \left| y_i^{(2)} - y_i^{(1)} \right|,$$

M>0. Then there is a segment $|x - x_0| \le \alpha$, on which functions exist $y_1, y_2, ..., y_n$, which are solutions of the normal system (1) and satisfy (1*).

Proof scheme. Similarly to the previous theorem, we first replace the normal system of differential equations with initial conditions by an equivalent system of integral equations:

$$\begin{cases} y_1(x) = y_{01} + \int_{x_0}^{x} f_1(t, y_1, y_2, ..., y_n)dt, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n(x) = y_{0n} + \int_{x_0}^{x} f_n(t, y_1, y_2, ..., y_n)dt. \end{cases}$$

Let's enter the operator $\psi = A\varphi$, which is written like this:

$$\psi_1(x) = y_{01} + \int_{x_0}^{x} f_1(t, \varphi_1, ..., \varphi_n)dt, \dots, \psi_n(x) = y_{0n} + \int_{x_0}^{x} f_n(t, \varphi_1, ..., \varphi_n)dt,$$

which set of functions $\varphi_1, ..., \varphi_n$ puts in accordance $\psi_1, ..., \psi_n$. A set of functions $(\varphi_1(x), ..., \varphi_n(x))$ defined on some segment $|x - x_0| \le \alpha$, belong to some subspace of continuous functions. Next, we apply Banach's theorem to the operator. To do this, we make sure that the three conditions of the theorem are fulfilled:

The subspace on which the functions are defined $\varphi_1, ..., \varphi_n$ complete, as a closed subspace of complete space.

1. We prove that there is a self-reflection here.
2. We prove that we have a compression operator.

Thus, according to Banach's theorem, there is a fixed point of this operator, that is, there is such a set of functions $(y_1(x), ..., y_n(x))$, whose image is the same set of functions $(y_1(x), ..., y_n(x))$, and this means that the system of integral equations has a unique solution, namely $(y_1(x), ..., y_n(x))$. Then the original system of differential equations also has a unique solution that satisfies the initial conditions.

Application of Banach's theorem to the solution of Fredholm's integral equations of the 2nd kind.

Fredholm's integral equation of the 2nd kind is called the equation of the form $\varphi(x)=\lambda\int_a^b K(x,t)\varphi(t)dt+f(x)$, where $K(x,t)$ – core, continuous in some closed rectangle $[a,b;c,d]$ function (known), $f(x)$ – free term, continuous function, $\lambda -$ parameter (real or complex number), $\varphi(x)$ – continuous unknown function.

Let's find it $\varphi(x)$ by the method of successive approximations. Consider the operator $\psi = A\varphi$.

$$\psi(x)=\lambda\int_a^b K(x,t)\varphi(t)dt+f(x),\ \varphi(x)\to\psi(x),\ \varphi(x)\subset C_{[a,b]}.$$

Let's check the conditions of Banach's theorem:

1) The space of continuous functions given by $[a,b]$ is complete.
2) We have a reflection in ourselves.

$$|\psi_2(x)-\psi_1(x)|=|\lambda|\left|\int_a^b K(x,t)(\varphi_2(t)-\varphi_1(t))dt\right|\le|\lambda|\int_a^b|K(x,t)||\varphi_2(t)-\varphi_1(t)|dt\le$$

$$\le|\lambda|M\int_a^b \max_t|\varphi_2(t)-\varphi_1(t)|dt=|\lambda|M\rho(\varphi_1,\varphi_2)|b-a|.$$

In that $K(x,t)$ is continuous in a closed rectangle, then according to the Weierstrass theorem it is bounded, i.e $|K(x,t)|\le M$. Inequality exists $\forall x\in[a,b]$, therefore also holds for the maximum. $\rho(\psi_1,\psi_2)\le|\lambda|M|b-a|\rho(\varphi_1,\varphi_2)$. Require that $|\lambda|M|b-a|<1$, so that the compression condition is fulfilled. Let's pick up λ, so that $|\lambda|<\dfrac{1}{M|b-a|}$.

Application of Banach's theorem to the solution of Hamerstein's nonlinear integral equations

This is an equation of the species $\varphi(x)=\lambda\int_a^b K(x,t,\varphi(t))dt+f(x)$. It is obvious that this equation is more complicated than the Fredholm equation, because the unknown function is included in the kernel. We will assume that $K(x,t,\varphi(t))$ – is continuous as a function of three variables (x,t,z) and satisfies the Lipshitz condition on z: $\left|K(x,t,z_2)-K(x,t,z_1)\right| \le M\left|z_2-z_1\right|$, $f(x)$– continuous. Consider the operator $\psi(x)=\lambda\int\limits_a^b K(x,t,\varphi(t))dt+f(x)$, $\psi=A\varphi$.

Let's check the fulfillment of the conditions of Banach's theorem:

1. $C_{[a,b]}$ – is complete.
2. Self reflection.
3. Compression:

$$\left|\psi_2(x)-\psi_1(x)\right|=\left|\lambda\right|\left|\int\limits_a^b K(x,t,\varphi_2)-K(x,t,\varphi_1)dt\right| \le \left|\lambda\right|\int\limits_a^b \left|K(x,t,\varphi_2)-K(x,t,\varphi_1)\right|dt \le$$

$$\le \left|\lambda\right| M\int\limits_a^b \left|\varphi_2(t)-\varphi_1(t)\right|dt \le \left|\lambda\right| M\left|b-a\right|\rho(\varphi_2,\varphi_1).$$

The inequality is valid for all x, so it also holds for the maximum, $\underbrace{\max_x \left|\psi_2(x)-\psi_1(x)\right|}_{\rho(\psi_2,\psi_1)} \le \left|\lambda\right| M\left|b-a\right|\rho(\varphi_2,\varphi_1)$. We require that the compression condition be fulfilled $\left|\lambda\right| < \dfrac{1}{M\left|b-a\right|}$. Therefore, the original Hamerstein integral equation has a unique solution.

Application of Banach's theorem to the solution of Voltaire's integral equations.

Voltaire's integral equation is called an equation of the form: $\varphi(x)=\lambda\int\limits_a^x K(x,t)\varphi(t)dt+f(x)$. We impose all the requirements on the functions as in the previous equations. Let us show that the method of successive approximations is applied for all x. We will show that the compression conditions are always fulfilled for the nth approximation at $n\to\infty$.

$$\left|\psi_2^{(1)}(x)-\psi_1^{(1)}(x)\right|=|\lambda|\left|\int_a^x K(x,t)(\varphi_2(t)-\varphi_1(t))dt\right|\le M|\lambda|\rho(\varphi_2,\varphi_1)(x-a),\ x>a;$$

$$\left|\psi_2^{(2)}(x)-\psi_1^{(2)}(x)\right|=|\lambda|\left|\int_a^x K(x,t)(\psi_2^{(1)}-\psi_1^{(1)})dt\right|\le M^2|\lambda|^2\rho(\varphi_2,\varphi_1)\cdot\frac{(x-a)^2}{2!};$$

....

$$\left|\psi_2^{(n)}(x)-\psi_1^{(n)}(x)\right|=|\lambda|\left|\int_a^x K(x,t)(\psi_2^{(n-1)}-\psi_1^{(n-1)})dt\right|\le M^n|\lambda|^n\rho(\varphi_2,\varphi_1)\cdot\frac{(x-a)^n}{n!}.$$

Having fixed $x=b$, we will have:

$\left|\psi_2^{(n)}(x)-\psi_1^{(n)}(x)\right|\le\frac{M^n|\lambda|^n\rho(\varphi_2,\varphi_1)}{n!}(b-a)^n$. You can always choose n so that $\frac{M^n|\lambda|^n(b-a)^n}{n!}<1$.

<u>Control questions</u>.

1. Define compressive mapping and prove its continuity.
2. Formulate and prove Banach's theorem. Give it a geometric interpretation.
3. Prove the theorem of existence and uniqueness of the solution of the Cauchy problem for the differential equation $y'=f(x,y)$, $f(x_0)=y_0$.
4. Give the definition of a normal system of differential equations.
5. Formulate a theorem on the existence and uniqueness of the solution of the Cauchy problem for a normal system of differential equations, indicate the scheme of its proof.
6. Using Banach's theorem, find out under what conditions there is a unique solution to Fredholm's integral equation.
7. Formulate the existence and uniqueness theorem of the solution of the nonlinear integral equation and the Voltaire equation.

Exercises.

1. Using Banach's theorem, find several approximations to the solution of the equation $x-\sqrt[4]{x+10}=0$ on a segment [1,2].
2. Operator A is given by relations

$$\begin{cases} y_1 = 0{,}1x_1 + 0{,}3x_2 - 0{,}2x_3 + 0{,}8; \\ y_2 = 0{,}2x_1 - 0{,}1x_2 \qquad\qquad -1{,}2; \\ y_3 = \qquad\quad -0{,}3x_2 + 0{,}2x_3 + 2{,}7. \end{cases}$$

Check the fulfillment of the conditions of Banach's theorem in space R_1^n.

3. How to estimate the error between the n-th approximation x_n of the solution of the equation $Ax=x$ and the exact value of x.
4. Find several approximations of the solution of the Cauchy problem $y'=y$, $y(0)=1$, and estimate the value of the segment on which the solution exists and is unique.

The solution.

1. The segment [1;2] is a complete space as a closed subspace of the complete space. Let's show that y=$\sqrt[4]{x+10}$ self-reflection, that is, its values do not go beyond the segment [1;2]. We have $f(1)=\sqrt[4]{11}>1$, $f(2)=\sqrt[4]{12}<2$. Furthermore $f'(x)=\dfrac{1}{4\sqrt[4]{(x+10)^3}}>0$, that is, the function is monotonic, and thus the function reaches its maximum and minimum values at the ends of the segment. We have that derivative $f'(x)=\dfrac{1}{4\sqrt[4]{(x+10)^3}}<1$ for $\forall\, x\in[1,2]$, that is, the compression condition is fulfilled. In this way, all three conditions of Banach's theorem are fulfilled and we can find the approximate value of the root of the equation by the method of successive approximations. For example, let's

put $x_0=1$. Then $x_1=\sqrt[4]{x+10}=\sqrt[4]{11}$, $x_2=\sqrt[4]{x_1+\sqrt{10}}=\sqrt[4]{\sqrt[4]{11}+\sqrt{10}}$ ...

2. First, space R_3 – complete. Secondly, we have a reflection in ourselves. Let's check the fulfillment of compression conditions:

$\alpha_1=\max\limits_k \sum\limits_{i=1}^{3}|a_{ik}|=\max(\sum\limits_{i=1}^{3}|a_{i1}|,\sum\limits_{i=1}^{3}|a_{i2}|,\sum\limits_{i=1}^{3}|a_{i3}|)$, where $\sum\limits_{i=1}^{3}|a_{ij}|$ – is the sum of the modules of the coefficients at x_j, which are respectively equal 0,3; 0,7; 0,4. Since the largest of the sums is equal to 0.7, then $\alpha_1=0{,}7<1$ and compression condition is fulfilled.

3. In the process of proving Banach's theorem, in particular, when proving the fundamentality of the sequence $\{x_n\}$, an estimate is obtained $\rho(x_n,x_m)\leq\frac{\alpha^n}{1-\alpha}\rho(x_0,x_1)$, where α – a constant that appears in the compression condition. Let's move on to the limit when m goes to infinity. We will get the desired grade: $\rho(x_n,x_m)\leq\frac{\alpha^n}{1-\alpha}\rho(x_0,x_1)$.

4. Differential equation $y'=y$ along with the initial condition $y(0)=1$, is equivalent to the following integral equation $y(x)=1+\int\limits_0^x y(t)dt$. Let's put $y_0=1$, then we will get: $y_1=1+\int\limits_0^x y_1 dt=1+x$;

$$y_2=1+\int\limits_0^x(1+t)dt=\left(1+\frac{(1+t)^2}{2}\right)\Bigg|_0^x=1+\frac{(1+x)^2}{2}-\frac{1}{2}=1+x+\frac{x^2}{2};\ldots$$

We note that the successive approximations coincide with the partial sums of the Taylor series for the function ex. Note that, as follows from Banach's theorem, the existence and uniqueness of the solution will be ensured in a rectangle centered at the point (0,1), $|x|<d, d<1, |y-1|<1$.

Problems.

1. Specify sufficient conditions that ensure the existence and unique root of the equation $F(x)=0$.
2. Show that when the function $f(x)$ of one variable has a derivative $f'(x)$, and $f'(x)\leq m<1$, then the mapping $y=f(x)$ will be compressive.
3. 3. Using Banach's theorem, find several approximate solutions of the equations:

а) $x^3+x-20=0$;

б) $x=\frac{1+\cos x}{10}$, $[0,1]$;

в) $x=(x+1)$;

г) $x=1-\frac{1}{x+2}$, $[0,4]$;

д) $x=x^3-2$, $[1,2]$;

е) $x=1+\frac{1}{2}\operatorname{arctg} x$;

є) $x=\sqrt{x+10}$, $[0,1]$;

ж) $x^2=\sin x$;

з) $x^2=\ln(x+1)$;

і) $x^3=e^x+2$.

4. Specify compression conditions in spaces R^n, R_0^n, R_1 to solve a system of n equations with n unknowns of the form:

$$\begin{cases} x_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1, \\ x_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = a_{n1}x_1 + a_{n1}x_2 + \ldots + a_{nn}x_n + b_n. \end{cases}$$

5. The equation is given: а) $y'=x-y^2$, $y(0)=0$;

б) $y'=x^2+y^2$, $y(0)=1$;

в) $y'=xy$, $y(0)=1$;

г) $y'=y^2+xy+x^2$, $y(0)=1$.

Find several approximate solutions of the given equations and estimate the value of the segment on which the solution exists and is unique.

COMPACTS

Let us give an example of a space for which the Bolzano-Weierstrass theorem does not hold, namely: a convergent subsequence can be isolated from any limited numerical sequence. This theorem is not valid in an arbitrary metric space. Consider space l_2: $x = (x_1, x_2, .., x_n, ...)$, $\sum_{k=1}^{\infty} x_k^2 < \infty$. Consider the sequence in this space:

(1,0,...,0,.................),
(0,1,0,...,0,...............),
....
(0,0,...,0,1,0,0,...,0,...),
....

This sequence will be bounded because all its members are in the middle of an abstract ball centered at a point 0 i $R = 1$. Really: $\rho(0, x^{(n)}) = \sqrt{\underbrace{(0-0)^2 + ... + 1^2}_{n} + (0-0)^2 + ...} = 1$. It is obvious that a convergent subsequence cannot be isolated from such a sequence, because if it were possible, the resulting sequence would be fundamental. But $\rho(x^{(n)}, x^{(m)}) = \sqrt{2}$ (does not go to zero). Let us distinguish the class of sets for which the Bolzano-Weierstrass theorem holds. This leads to the concept of compactness.

Definition. Let us have a metric space (X, ρ) and a set $E \subset (X, \rho)$. A set E is called *compact* if from any sequence of this set it is possible to isolate a convergent subsequence whose boundary belongs to the set E.

Definition. A set E is called *compact*, if a finite subcover can be isolated from any of its infinite covers by a system of open sets.

Examples of compacts.

1. Segment $[a,b]$ – compact.
2. Interval $(0,1)$ – is not compact.

Theorem. Every compact set is a closed and bounded set.

Proof. Closed-mindedness Let x be the boundary point of E. We will prove that it belongs to E. Consider the system of abstract balls $B_1(x,1),\ B_2(x,\frac{1}{2}),\dots,\ B_n(x,\frac{1}{n}),\dots$ with center at x and radii $\frac{1}{n}$. Then we will choose a representative from each ball $x_1,x_2,\dots,x_n,\dots$. $\{x_n\}\subset E$; $\lim\limits_{n\to\infty} x_n = x$; $\rho(x,x_n)<\frac{1}{n}\to 0$ at $n\to\infty$. Therefore, any subsequence also goes to x. And according to the definition of the compact, E will belong, and this proves closedness.

Limitation. Let the compact set E be an unbounded set. Let's take an arbitrary ball $B(x_0,r)$, де x_0 – is an arbitrary point of the compact, r is an arbitrary number. Due to not limited, there will be a point x_1 that does not belong to the ball $B(x_0,r_1)$ with a radius $r_1=\rho(x_0,x_1)+1$. Due to not limited, there will be a point x_2 that does not belong to the ball $B(x_0,r_2)$ with a radius $r_2=\rho(x_0,x_2)+1$. Continuing this process, we will get a sequence of elements: $x_0,x_1,\dots,x_n\dots$. This sequence will not be fundamental, because the distance between any of its elements will be greater than 1. Moreover, any subsequence will also not be fundamental, which contradicts the fact that E is compact.

The theorem is proved.

Compact in *R^m*.

Compact in R^m can be described by a theorem.

Theorem. In order to set $E\subset R^m$ was compact, it is necessary and sufficient that it be closed and limited.

Proof. Necessity. Given that E is a compact set, then from the previous theorem the set is closed and limited.

Sufficiency. Let E be closed and limited. Let us prove that E is compact. Consider an arbitrary sequence $\{x_n\}\subset E$. It will be limited in the sense that all its points will be located in a ball with the center at an arbitrary point x and a finite radius R. According to the Bolzano-Weierstrass theorem from the sequence $\{x_n\}$ a convergent subsequence can be distinguished. Let $\lim\limits_{n\to\infty} x_n = x_0$. Let's show that $x_0 \in E$. Since x_0 – limit point, then due to closedness $x_0 \in E$.

The theorem is proved.

Properties of continuous mappings of compacts.

Theorem 1. The continuous image of a compact is a compact.

Proof. It is given that E is compact, $f(x)$ – continuous mapping. Consider an arbitrary sequence $y_1,...,y_n$, which belongs $f(E)$ and we will show that it is possible to isolate a convergent subsequence whose limit belongs to the set of images. Consider the corresponding sequence of prototypes $x_1,...,x_n$. It belongs to E, and therefore a convergent subsequence can be extracted from it $\{x_{n_k}\}$, such that $\lim\limits_{k\to\infty} x_{n_k} = x$, $x \in E$. Then due to the continuity of the mapping $\lim\limits_{k\to\infty} f(x_{n_k}) = f(x)$, $f(x) \in f(E)$, because x belongs to the set of proimages.

Theorem 2. Let us have a continuous and mutually unique mapping $y = f(x)$ of the compact set E on the set Y. Then the inverse mapping $x = f^{-1}(y)$ is also continuous.

Proof. Suppose to the contrary that the inverse map is not continuous at an arbitrary point y. Then, according to Heine's definition of continuity, there will be a sequence $\{y_n\} \to y$, but, $\{x_n = f^{-1}(y_n)\}$ does not go to $f^{-1}(y)$. This means that two cases are possible:

1. $x_n \to \bar{x} \neq x$;
2. $\lim\limits_{n\to\infty} x_n$ – does not exist.

1. Due to the continuity of direct mapping $f(x_n) \to f(\bar{x})$, but on condition

$f(x_n)\to f(x)$, then we have that the sequence has two limits, which is impossible.

2. $\{x_n\}\subset E$, then from this sequence it is possible to distinguish a convergent subsequence of the limit to which E belongs, because E is compact. So will exist $\{x_{n_k}\}\to \bar{x}\neq x$. Then by virtue of continuity $f(x_{n_k})\to f(\bar{x})$, and this contradicts the fact that $\{y_n\}\to y$. Because of $\{y_n\}\to y$, then any subsequence $\{y_{n_k}\}\to y$. We came to a contradiction.

The theorem is proved.

Uniformly continuous mappings. Cantor's theorem.

Definition. Mapping *f(x)* is called uniformly continuous on the set *E* if for $\forall\varepsilon>0\ \ \exists\delta>0$: $\rho(x'',x')<\delta$, $\rho(f(x''),f(x'))<\varepsilon$.

Theorem (Cantor's). If the mapping $y=f(x)$ – is continuous on the compact set *E*, then it is uniformly continuous on it.

Proof. Let the mapping not be uniformly continuous. Then $\exists\varepsilon_0$, that which δ we did not take, there will be at least two elements x'', x', that as soon as $\rho(x'',x')<\delta$, then $\rho(f(x''),f(x'))\geq\varepsilon_0$. Due to arbitrariness δ, let's choose it like this: $\delta_n=\frac{1}{n}$, $n=1,2,....$. For everyone δ there will be a pair of points x'_n *and* x''_n, that as soon as $\rho(x''_n,x'_n)<\frac{1}{n}$, than $\rho(f(x'),f(x''))\geq\varepsilon_0$ (*).Two sequences were obtained $\{x'_n\},\{x''_n\}$, which belong to *E*. Then from the sequence $\{x'_n\}$ it is possible to distinguish a convergent subsequence with a limit x_0. In order not to introduce new indexes, we will assume that $\{x'_k\}\to x_0$. Let's show that $\{x''_n\}\to x_0$.

Really: $\rho(x''_n,x_0)\leq\underbrace{\rho(x''_n,x'_k)}_{\Rightarrow 0}+\underbrace{\rho(x'_k,x_0)}_{\Rightarrow 0}$. Then due to the continuity of the reflection $f(x_n')\to f(x_0)$ i $f(x_n'')\to f(x_0)$, and this contradicts (*),because:

$$\rho(f(x_n''),f(x_n'))\le\underbrace{\rho(f(x_n''),f(x_0))}_{=0}+\underbrace{\rho(f(x_0),f(x_n'))}_{=0}.$$

The theorem is proved.

Numerical functions on a compact.

Mapping $y=f(x)$, $x\in E$, which corresponds to an element of arbitrary nature *x* with a number is called a functional.

The first theorem of Weierstrass. If a numeric function $y=f(x)$ is continuous on the compact set *E*, then it is limited on it.

Proof. The continuous image of a compact is a compact, and a compact is a limit set, which proves the theorem.

The second theorem of Weierstrass. If a numeric function $y=f(x)$ is continuous on the compact set *E*, then it reaches its largest and smallest value on it.

Proof. According to the previous theorem $\exists m\ i\ M$, that $m\le f(x)\le M$. If a function is limited from below and above, then it has its exact upper and lower faces, $m^*\le f(x)\le M^*$. Let us prove for definiteness that *f(x)* achieves its $\sup M^*$, where M^*– is a limit point *f(E)*, because in any neighborhood of it there is an infinite number of points from *f(E)*. And since *f(E)* the function is closed, then $M^*\in f(E)$.

***Control questions*.**

1. Give an example of a space in which the Bolzano-Weierstrass theorem does not hold.
2. Give the definition of a compact.
3. Formulate the main properties of compacts.

4. To characterize compacts in space R^n.
5. Indicate the scheme of proving the theorem on the compactness of an image in the case of a continuous representation.
6. Formulate the theorem on the continuity of the inverse mapping.
7. Formulate the first and second theorems of Weierstrass and Cantor's theorem on the properties of numerical functions on a compact.

Exercises.

1. Investigate the compactness of the space of isolated points.
2. Prove that the segment $[a,b]$ is compact.
3. Prove the first theorem of Weierstrass about the boundedness of a continuous numerical function on a compact set using compactness properties.
4. Check the compactness of the interval (0,1).

The solution.

1. Consider an arbitrary sequence from the space of isolated points. Since the sequence is infinite, one or more points will repeat. Then it is clear how a convergent subsequence can be isolated from such a sequence.
2. Take an arbitrary numerical subsequence that belongs to a segment $[a,b]$. According to the Bolzano-Weierstrass theorem, a convergent subsequence can be isolated from it, the boundary of which, due to the closedness of the segment, belongs to $[a,b]$.
3. Since the continuous image of a compact is a compact, and a compact is a bounded set, it follows from this that the set of values of a continuous function *f(x),* $x \in [a,b]$ will be bounded, which proves the first theorem of Weierstrass.

4. Interval (0,1) is not compact, because it is a sequence $\left(\frac{1}{n}\right)$ (hence any subsequence of it) goes to zero when $n \to \infty$, and zero does not belong to the interval (0,1).

Problems.

1. Investigate the compactness of a set of rational and irrational points of a segment [0,1].
2. Prove that the unit square of space is closed R_2 there is a compact.
3. Prove that every compact set is a closed and bounded set.
4. Prove that the intersection of two compact sets is a compact set.
5. Prove that the union of two compact sets is a compact set.
6. Prove the first and second theorems of Weierstrass on the properties of numerical functions on a compact set, using the fact that a compact set is a closed and bounded set.

HEINE-BORELL'S LEMMA

Let's consider together with an interval $[a,b]$ some system $\sum$ open segments σ, which can be both finite and infinite. Let's agree that the system $\sum$ covers the interval $[a,b]$, if for each point $x \in [a,b]$ will be found $\sum$ in the interval σ, containing it.

Borel's Lemma. If the interval is closed $[a,b]$ is covered by an infinite system $\sum = \{\sigma\}$ open segments, then a finite subsystem can always be isolated from it $\sum^* = \{\sigma_1, \sigma_2, ..., \sigma_n\}$, which will also cover the interval $[a,b]$.

Proof 1 (The Bolzano method).

We will carry out the proof by the method of the opposite. Assume an interval $[a,b]$ cannot be covered by a finite number of segments σ of $\sum$. Let's divide the interval $[a,b]$ in half. Then at least one of its halves cannot be covered by a finite number either σ. Indeed, if one of them could be covered with segments $\sigma_1,\sigma_2,...,\sigma_m$ (із $\sum$), and the second - in segments $\sigma_{m+1},\sigma_{m+2},...,\sigma_n$ (із $\sum$), then all these segments would form a finite system $\sum^*$, which would cover the entire gap $[a,b]$. And this contradicts the assumption. Denote through $[a_1,b_1]$ that half of the segment that is not covered by a finite number σ (if both are such, then any of them). We will divide this interval in half again and mark it through $[a_2,b_2]$ the one of its halves that cannot be covered by a finite number σ, etc.

Continuing this process, we obtain an infinite sequence of nested segments $[a_n,b_n]$ ($n=1,2,3,...$),each of which is half of the previous one. All these intervals are chosen so that none of them is covered by a finite number of segments σ. According to the lemma about nested segments, there is a point *c* common to all of them, to which the ends go $[a_n,b_n]$.

This point *c* is like any point in the interval $[a,b]$, lies in one of the segments σ, which we denote $\sigma_0=(\alpha,\beta)$, so $\alpha<c<\beta$. However, the sequence $\{a_{\text{n}}\}$ and $\{b_{\text{n}}\}$ are heading to *c*, and starting from a certain number they will lie between themselves α and β, in such a way that the interval defined by them $[a_n,b_n]$ will be covered by only one segment σ_0, contrary to the very choice of these intervals $[a_n,b_n]$. The obtained contradiction proves the lemma.

<u>Proof 2 (Lebesgue)</u>

Consider the points $x^*\in[a,b]$, which have the same properties as a segment $[a,x^*]$ is covered by a finite number of segments σ. Such points x^*, in general, there will be: since, for example, the point *a* lies in one of σ, then all points close to it that are in this segment σ, are shown as points, respectively x^*.

Our task is to establish that point b belongs to the number of points x^*.

As all $x^* \le b$, that: $\sup\{x^*\} = c \le b$. Like all points from the gap $[a,b]$, c belongs to some $\sigma_0 = (\alpha, \beta)$, $\alpha < c < \beta$. However, according to the properties of the exact upper face, will be found x_0^*, so what $\alpha < x_0^* \le c$. Interval $[a, x_0^*]$ is covered by a finite number of segments σ (according to the definition of points x^*), if one more segment is added to these intervals σ_0, then the whole gap will be covered $[a,c]$, so that c is one of the points x^*.

Along with this, it is clear that c cannot be less than b, because otherwise between c and β there would be another point x^*, contrary to finding the number c as the upper limit of all x^*. Thus, it is necessary that b=c, so b is one of x^*, that is $[a,b]$ it is covered by a finite number of segments σ.

Note that the lemma holds when $[a,b]$ – is closed and $\sum$ – is open. For example, a system of open segments: $\left(\frac{1}{2},\frac{3}{2}\right),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{8},\frac{3}{8}\right),\ldots,\left(\frac{1}{2^n},\frac{3}{2^n}\right),\ldots$ covers the interval $(0,1]$, however, a finite subsystem with the above-mentioned properties cannot be isolated from them. Similarly, the system of closed segments: $\left[0,\frac{1}{2}\right],\left[\frac{1}{2},\frac{3}{4}\right],\left[\frac{3}{4},\frac{7}{8}\right],\ldots,\left[\frac{2^{n-1}}{2^n},\frac{2^{n+1}-1}{2^{n+1}}\right],\ldots$ and $[1,2]$, covers the interval $[0,2]$, but even here the selection of a finite subsystem is impossible.

New proofs of basic theorems.

Let us now show how Borel's Lemma can be used to prove basic theorems about continuous functions.

I. <u>The first Bolzano-Cauchy theorem.</u> Let the function $f(x)$ be continuous on the segment $[a,b]$ and takes the values of different signs at the ends of the segment. Then in the interval (a,b) there is a point where the value of the function is zero, i.e $f(a)f(b)<0 \Rightarrow \exists\xi \in (a,b): f(\xi)=0$.

Proof

This time we will prove it from the opposite side. Let us assume that - subject to the assumption of the theorem - the function is still at no point $f(x)$ does not become zero. Then every point x' in the interval $[a,b]$ can be limited to such an area $\sigma' = (x'-\delta', x'+\delta')$, that is within its limits $f(x)$ preserves a certain sign.

An infinite system $\sum = \{\sigma\}$ of these circles covers the entire interval $[a,b]$. Then, according to Borel's lemma, will exist $\sum^*$ there be a circle that covers this segment ($\sum^*$ – finite).

The left end *a* of our interval belongs to one of the neighborhoods of the system $\sum^*$, let around $\sigma_1 = (x_1-\delta_1, x_1+\delta_1)$. Its right end $x_1+\delta_1$, in turn, belongs to the circle $\sigma_2 = (x_2-\delta_2, x_2+\delta_2)$ from $\sum^*$, point $x_2+\delta_2$ contained in the neighborhood (interval) $\sigma_3 = (x_3-\delta_3, x_3+\delta_3)$ from $\sum^*$, etc.

After a finite number of steps, moving to the right, we will reach approx $\sigma_n = (x_n-\delta_n, x_n+\delta_n)$ from $\sum^*$, which contains the right end *b* of this interval. If $\sum^*$ contained some other gaps, except $\sigma_1, \sigma_2, ..., \sigma_n$, then, obviously, they could simply not be taken into account.

In the neighborhood σ_1 function $f(x)$ preserves a certain sign, the sign itself $f(a)$. However, in σ_2 the function has a defined sign, which must also match the sign $f(a)$, as σ_1 i σ_2 overlap each other. We also make sure that this sign is preserved by the function in the next neighborhood as well σ_3, which overlaps with σ_2, etc. In the end, let's come to the conclusion that in the last neighborhood σ_n the function has a sign $f(a)$, so $f(b)$ coincides in sign with $f(a)$. Got a contradiction.

The theorem is proved.

An intermediate value of a function that is continuous on a segment. Dichotomy method.

Suppose $f(a)<0$, $f(b)>0$. Let's divide the segment $[a,b]$ in half $\frac{a+b}{2}$. Three cases are possible:

1. $f\left(\frac{a+b}{2}\right)=0 \Rightarrow \xi=\frac{a+b}{2}$, that is, the theorem is proved.

2. $f\left(\frac{a+b}{2}\right)>0$. Let's put $a_1=a$, $b_1=\frac{a+b}{2}$.

3. $f\left(\frac{a+b}{2}\right)<0$. Let's put $a_1=\frac{a+b}{2}$, $b_1=b$.

We will get $[a,b]\supset[a_1,b_1], f(a_1)<0, f(b_1)>0$. Let's divide the obtained segment again by a point $\frac{a_1+b_1}{2}$. Again, three cases are possible:

1. $f\left(\frac{a_1+b_1}{2}\right)=0 \Rightarrow \xi=\frac{a_1+b_1}{2}$, that is, the theorem is proved.

2. $f\left(\frac{a_1+b_1}{2}\right)>0$. Let's put $a_2=a_1$, $b_2=\frac{a_1+b_1}{2}$.

3. $f\left(\frac{a_1+b_1}{2}\right)<0$. Let's put $a_2=\frac{a_1+b_1}{2}$, $b_2=b_1$.

We will get: $[a_1,b_1]\supset[a_2,b_2]$, $f(a_2)<0, f(b_2)>0$.

Let's continue the process of building such segments. On $(n-1)$ steps we have:

$[a,b]\supset[a_1,b_1]\supset[a_2,b_2]\supset...\supset[a_{n-1},b_{n-1}], f(a_{n-1})<0$, $f(b_{n-1})>0$.

Let's divide this segment $[a_{n-1},b_{n-1}]$ into 2 parts with a point $\frac{a_{n-1}+b_{n-1}}{2}$. Again we have three cases:

1. $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right)=0 \Rightarrow \xi=\frac{a_{n-1}+b_{n-1}}{2}$, that is, the theorem is proved.

2. $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right)>0$. Let's put $a_n = a_{n-1}$, $b_n = \frac{a_{n-1}+b_{n-1}}{2}$

3. $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right)<0$. Let's put $a_n = \frac{a_{n-1}+b_{n-1}}{2}$, $b_n = b_{n-1}$, we will get

$[a,b]\supset[a_1,b_1]\supset[a_2,b_2]\supset...\supset[a_n,b_n], f(a_n)<0, f(b_n)>0$.

The length of the n-th segment $b_n - a_n = \frac{b-a}{2^n} \to 0$,

$a_n \to \xi$, $b_n \to \xi$ at $n\to\infty$ (according to the lemma about nested segments). $f(a_n)\to f(\xi)$, $f(b_n)\to f(\xi)$, due to the continuity of the function.

Simultaneously $f(\xi)\le 0$ (since $f(a_n)<0$) i $f(\xi)\ge 0$ (since $f(b_n)>0$), namely $f(\xi)=0$.

The theorem is proved.

The geometric meaning of the Bolzano-Cauchy theorem is that a continuous function crosses the Ox axis at least once.

II. The first theorem of Weierstrass. If the function $f(x)$ continuous on the segment $[a,b]$, then it is limited on this segment, i.e $\exists M, m$: $m\le f(x)\le M$.

Proof.

Due to the continuity of the function $f(x)$, whatever point $x$we did not take, there will be such a number $\varepsilon>0$, that can be selected as arbitrarily small around this point $\sigma'=(x'-\delta', x'+\delta')$, so that for all its values of x the inequalities would be satisfied:

$|f(x)-f(x')|<\varepsilon$, or $f(x')-\varepsilon<f(x)<f(x')+\varepsilon$.

Thus, within each such circle is a function $f(x)$ limited in advance: from below - by a number $f(x')-\varepsilon$, and above - with a number $f(x')+\varepsilon$.

It is clear that here Borel's lemma should be applied to an infinite system $\sum$ of circles that possess the specified property. It follows that what will be found in $\sum$ finite number of rounds $\sigma_1,\sigma_2,...,\sigma_n$, which together will cover the entire segment $[a,b]$. If:

$m_1 \le f(x) \le M_1$ в σ_1,

$m_2 \le f(x) \le M_2$ в σ_2,

……………………..

$m_n \le f(x) \le M_n$ в σ_n,

Having put $m = \min\{m_1, m_2, ..., m_n\}$ i $M = \max\{M_1, M_2, ..., M_n\}$. Obviously we will have $m \le f(x) \le M$ throughout the interval $[a,b]$.

The theorem is proved.

III. Cantor's theorem. If the function $f(x)$ is continuous on a segment $[a,b]$, then it is uniformly continuous on this segment.

Proof.

Let's set an arbitrary number $\varepsilon > 0$. This time every point x' segment $[a,b]$ let's limit it to such an area $\sigma' = (x' - \sigma', x' + \sigma')$, so that inequality is fulfilled within its limits: $|f(x) - f(x')| < \frac{\varepsilon}{2}$.

If x_0 is also a point of this circle, then at the same time and $|f(x') - f(x_0)| < \frac{\varepsilon}{2}$. Thus, for any points x and x_0 from σ' we will have: $|f(x) - f(x_0)| < \varepsilon$.

Let's download each circle σ' twice, keeping its center, i.e. instead of σ' consider the circle: $\overline{\sigma'} = \left(x' - \frac{\delta'}{2}, x' + \frac{\delta'}{2} \right)$.

A system will be formed from these neighborhoods accordingly $\overline{\Sigma}$, which covers the segment $[a,b]$, and it is to her that we will apply Borel's lemma. Interval $[a,b]$ will be covered by a finite number of segments from $\overline{\Sigma}$:

$\overline{\sigma_i} = \left(x_i - \frac{\delta_i}{2}, x_i + \frac{\delta_i}{2} \right)$ $(i = 1,2,...,n)$.

Let now δ will be the smallest of all these numbers $\frac{\delta_i}{2}$, i x_0, x – any two points of our interval that satisfy the condition:
$|x - x_0| < \delta$. (*)

The point x_0 must belong to one of the selected circles, for example, circle:
$\overline{\sigma_{i_0}} = \left(x_{i_0} - \frac{\delta_{i_0}}{2}, x_{i_0} + \frac{\delta_{i_0}}{2} \right)$, that $\left|x_0 - x_{i_0}\right| < \frac{\delta_{i_0}}{2}$.

As $\delta \le \frac{\delta_{i_0}}{2}$, then, according to (*), $|x - x_0| < \frac{\delta_{i_0}}{2}$, from $|x - x_i| < \delta_{i_0}$, i.e., the point x (especially the point x_0) belongs to that initially taken circle: $(x_{i_0} - \delta_{i_0}, x_{i_0} + \delta_{i_0})$, by compressing which we get approx $\overline{\sigma_{i_0}}$. In this case, due to the properties of the circles taken earlier, we get: $|f(x) - f(x_0)| < \varepsilon$.

As δ regardless of the position of the point x_0, uniform continuity of the function was chosen $f(x)$ is proved.

The theorem is proved.

As can be seen from the above considerations, Borel's lemma is successfully applied in those cases when the "local" property associated with the vicinity of a single point is subject to distribution over the entire considered interval.

***<u>Control questions</u>*.**

1. Formulate and prove the Heine-Borel lemma.
2. 2. Give the definition of compactness of a set due to coverage.
3. Prove the first Bolzano-Cauchy theorem on the existence of a root of equations $f(x) = 0$ using the Heine-Borel lemma.
4. Formulate the Heine-Borel lemma in an arbitrary abstract space.

5. Using the Heine-Borel lemma, prove Cantor's theorem on the uniform continuity of a numerical function $f(x)$, which is continuous on the segment $[a,b]$.

Exercises.

1. Using the Heine-Borel lemma, prove the first theorem of Weierstrass about the boundedness of a numerical function that is continuous on a segment $[a,b]$.

The solution.

1. Since the function $f(x)$ continuous, then any point x' segment $[a,b]$ we didn't take it by giving a number $\varepsilon>0$, we can take a fairly small neighborhood $\sigma'=(x'-\delta',x'+\delta')$ such that the inequalities hold for all points around x $|f(x)-f(x')|<\varepsilon$ or $f(x')-\varepsilon<f(x)<f(x')+\varepsilon$. The set of these circles forms an infinite covering of a segment $[a,b]$. Within each such circle is a function $f(x)$ is limited in advance: from below - by a number $f(x')-\varepsilon$, and above - with a number $f(x')+\varepsilon$. According to the Heine-Borel lemma, a finite subcover can be extracted from this infinite cover. If

$$
\begin{gathered}
m_1 \le f(x) \le M_1 \ \text{в} \ \sigma_1, \\
m_2 \le f(x) \le M_2 \ \text{в} \ \sigma_2, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
m_n \le f(x) \le M_n \ \text{в} \ \sigma_n,
\end{gathered}
$$

then taking m as the smallest of the numbers $m_1, m_2, \ldots, m_n$, and as M is the largest number $M_1, M_2, \ldots, M_n$, obviously, we will have

$$m \le f(x) \le M$$

on the entire segment $[a,b]$, which had to be proved.

LINEAR, NORMAL AND EUCLIDEAN SPACES

Linear spaces.

The concept of linear space is one of the basic ones in functional analysis.

<u>Definition.</u> A set of elements of an arbitrary nature $\{x, y, z, ...\}$ is called a *linear space* if:

1. an element addition operation is defined on this set, which matches two elements x and y with a third element z, which is called a sum. $\forall x, y \to z$, $z = x + y$ and properties are performed:

а) $x + y = y + x$ – commutativity;

б) $x + (y + z) = (x + y) + z$ – associativity;

в) $\exists 0,\ x + 0 = x$;

г) $\exists \bar{x},\ \bar{x} + x = 0$;

2. the operation of multiplication by a number (real or complex) $\alpha \cdot x = \alpha x$ and properties are performed:

а) $\alpha(\beta x) = (\alpha\beta)x$; – associativity;

б) $\exists 1,\ 1 \cdot x = x$;

в) $(\alpha + \beta)x = \alpha x + \beta x$ – distributive law;

г) $\alpha(x + y) = \alpha x + \alpha y$ – distributive law.

<u>Examples.</u>

1. R^1 – linear space.
2. R^n: $x = (x_1, x_2, ..., x_n)$, $y = (y_1, ..., y_n)$,

   $x + y = (x_1 + y_1, ..., x_n + y_n)$, $x \cdot \alpha = (\alpha x_1, ..., \alpha x_n)$.
3. $C_{[a,b]}$: $x = x(t),\ y = y(t)$; $x + y = x(t) + y(t)$; $\alpha \cdot x(t) = \alpha x(t)$.

Let's introduce the concepts of linear dependence and linear independence of elements.

<u>Definition.</u> Elements of arbitrary nature $x_1, x_2, ..., x_n$ are called *linearly dependent* if there is a set of numbers $\alpha_1, \alpha_2, ..., \alpha_n$, among which at least

one is different from zero, that equality holds $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$.

The fact of linear dependence means that one of the elements can be expressed as a linear combination of other elements:

$$\alpha_1 \neq 0, \qquad x_1 = -\frac{\alpha_2}{\alpha_1} x_2 - \frac{\alpha_3}{\alpha_1} x_3 - \ldots - \frac{\alpha_n}{\alpha_1} x_n .$$

Definition. Elements $x_1, \ldots, x_n$ are called linearly independent if equality $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$, holds if and only if all $\alpha_i = 0$, $i = \overline{1, n}$.

Example.

Investigate the linear dependence of functions:

1) $\sin^2 x$, $\cos^2 x$, 5 $(-\infty, \infty)$. We have: $1 \sin^2 x + 1 \cos^2 x - \frac{1}{5} \cdot 5 = 0$, that is, the functions are linearly dependent.

2) $1, x, x^2, \ldots, x^n$ $(-\infty, \infty)$; $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n = 0$; Independent.

In the left part, an algebraic polynomial of the n-th degree. It has $m \leq n$ roots, and we need it to be zero for any *x*. This is possible only when everyone $\alpha_i = 0$.

Definition. A linear space is called *n-dimensional* if there are *n* linearly independent elements in it, and any $(n+1)$elements are already linearly dependent.

These n linearly independent elements are called the basis of this space.

$$R^n : \begin{array}{c} (1,0,0,\ldots,0) \\ (0,1,0,\ldots,0) \\ \ldots\ldots\ldots\ldots \\ (0,0,0,\ldots,1) \end{array} \quad \text{– basis.}$$

Definition. A linear space is called *infinitely dimensional* if there is an arbitrarily large number of linearly independent elements in it. Finite-dimensional spaces are studied in linear algebra, and infinite-dimensional spaces in functional analysis.

1. $C_{[a,b]}$: $x = x(t)$, $1, x, x^2, \ldots, x^n \ldots$ – basis

2. l_2 : $x=(x_1,x_2,...,x_n,...)$, $\sum_{k=1}^{\infty} x_k^2 < \infty$; $\begin{pmatrix} (1,0,..........) \\ (0,1,..........) \\ (0,...,0,1,....) \\ .... \end{pmatrix}$ – basis.

<u>Definition.</u> A set of elements from the space *L* is called *subspace* L^* of a linear space *L*, if from the fact that x and $y \in L^*$ it follows that $(\alpha x + \beta y) \in L^*$, with everyone α and β.

Every space *L* contains a subspace of itself and the null space. A subspace that does not coincide with *L* and the zero subspace is called an eigensubspace *L*.

Let us have some set $\{x_k\} \subset L$. The linear envelope of a given set is the smallest linear subspace of the space *L* that contains the elements of the set $\{x_k\}$.

Normed spaces.

Functional analysis deals with spaces that are both linear and metric. Normed spaces belong to such spaces. Let us have a linear space *L*.

<u>Definition.</u> A space *L* is called *Normed* if every element $x \in L$ matched number $\|x\|$ – norm x that satisfies the axioms:

1. $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\forall \alpha \in R, C$, $\|\alpha x\| = |\alpha| \cdot \|x\|$, *R* – real, *C*– complex;
3. $\|x + y\| \leq \|x\| + \|y\|$.

Interest in these spaces is caused by the fact that they are easy to meter, that is, to introduce the concept of distance $\rho(x, y) = \|x - y\|$.

1. $\rho(x, y) \geq 0$; $\rho(x, y) = 0 \Leftrightarrow x \equiv y$;
2. $\rho(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1| \cdot \|y - x\| = \rho(y, x)$;
3. $\rho(x, y) = \|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|z - y\| = \rho(x, z) + \rho(z, y)$.

The norm of an element *x* is the distance between it and the zero element $\|x\| = \|x - 0\| = \rho(x,0)$.

Examples.

1. R^n, $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$;

2. l_2, $\|x\| = \sqrt{\sum_{k=1}^{\infty} x_k^2}$, $x = (x_1, x_2, \ldots, x_n, \ldots)$, $\sum_{k=1}^{\infty} x_k^2 < \infty$ – збіжний;

3. $C_{[a,b]}$, $x = x(t)$, $\|x\| = \max_t |x(t)|$.

Definition. Complete normed spaces are called Banach spaces.

Euclidean spaces.

Consider space R^n, its elements are sets of real numbers $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$.

The scalar product of two vectors is called a number $(x, y) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$.

In an arbitrary linear abstract space, the concept of a scalar product is defined axiomatically. Let us have some linear space L.

Definition. Numeric (x, y) is called a *scalar product* of two elements if it satisfies the following axioms:

1. $(x, y) = (y, x)$;
2. $(\lambda x, y) = \lambda (x, y)$, λ – numeric;
3. $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$;
4. $(x, x) \geq 0$, $(x, x) = 0 \Leftrightarrow x = 0$.

Definition. A linear space with a scalar product fixed in it is called Euclidean.

Examples.

1. l_2, $(x, y) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n + \ldots$. The convergence of the series follows from an obvious inequality $|x_i y_i| \leq \frac{1}{2}(x_i^2 + y_i^2)$.

2. $C_{[a,b]}$, $x = x(t)$, $(x, y) = \int_a^b x(t) y(t) dt$.

The fulfillment of all axioms of the scalar product follows from the properties of the definite integral.

1. $(x,y)=\int_a^b x(t)y(t)dt=\int_a^b y(t)x(t)dt=(y,x)$;

2. $(\lambda x,y)=\int_a^b \lambda x(t)y(t)dt=\lambda\int_a^b x(t)y(t)dt=\lambda(x,y)$;

3. $(x_1+x_2,y)=\int_a^b (x_1(t)+x_2(t))y(t)dt=\int_a^b x_1(t)y(t)dt+\int_a^b x_2(t)y(t)dt=(x_1,y)+(x_2,y)$;.

4. $(x,x)=\int_a^b x^2(t)dt\ge 0$. If $x(t)=0$, then $\int_a^b 0dt=0$. Let $\int_a^b x^2(t)dt=0$, and let's prove that $x(t)=0$. Let there exist a point t_0, in which $x(t_0)>0$, then, by virtue of continuity, the function $x(t)$ will be greater than zero in some neighborhood as well t_0. Then $\int_a^b x^2(t)dt\ne 0$. We got a contradiction.

Euclidean space can be normalized by putting $\|x\|=\sqrt{(x,x)}$. To prove this, we first prove the Cauchy-Buniakovsky lemma.

Lemma. There is an inequality in Euclidean space: $|(x,y)|\le\|x\|\cdot\|y\|$.

Proof. Let's use the function $\rho(\lambda)=(\lambda x+y,\lambda x+y)\ge 0$,

$$\rho(\lambda)=(\lambda x,\lambda x+y)+(y,\lambda x+y)=(\lambda x,\lambda x)+(\lambda x,y)+(y,\lambda x)+(y,y)=$$
$$=\lambda^2\|x\|^2+2\lambda(x,y)+\|y\|^2\ge 0.$$

A quadratic trinomial with respect to the parameter was obtained λ.

$D=4(x,y)^2-4\|x\|^2\cdot\|y\|^2\le 0$; $(x,y)^2\le\|x\|^2\cdot\|y\|^2$; $|(x,y)|\le\|x\|\cdot\|y\|$.

Having considered specific spaces R^n and $C_{[a,b]}$, and finding their scalar product, x and y norms, we will get two classical Cauchy-Buniakovsky inequalities known to us. Next, we will check the fulfillment of the axioms of the norms.

1. $\|x\|=\sqrt{(x,x)}\ge 0$, $\|x\|=0\Leftrightarrow x=0$;

2. $\|\lambda x\| = \sqrt{(\lambda x, \lambda x)} = \sqrt{\lambda^2 (x,x)} = |\lambda| \sqrt{(x,x)} = |\lambda| \cdot \|x\|$;

3. $\|x+y\|^2 = (x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y) = \|x\|^2 + 2(x,y) + \|y\|^2 \le$

$\le \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$.

Having obtained the square root of both parts, we get: $\|x+y\|^2 \le \|x\| + \|y\|$. The Cauchy-Buniakovsky inequality allows us to introduce the concept of the cosine of the angle between two abstract elements $\cos\varphi = \dfrac{(x,y)}{\|x\| \cdot \|y\|}$.

Two elements x and y from the Euclidean space will be considered orthogonal if their scalar product is equal to zero.

Definition. System of elements $\{x_\alpha\}$ from the Euclidean space is called orthogonal if: $(x_\alpha, x_\beta) = 0$ at $\alpha \ne \beta$; $(x_\alpha, x_\beta) \ne 0$ at $\alpha = \beta$. If $(x_\alpha, x_\alpha) = 1$, then such a system is called orthonormal.

It can be proved that when the system is orthonormal, it is linearly independent, that is, that $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0$ if and only if $\alpha_i = 0$. Indeed, consider the scalar product

$(x_i \alpha_i, x_1 \alpha_1 + x_2 \alpha_2 + ... + x_n \alpha_n) = 0$,

$\underbrace{(x_i \alpha_i, x_1 \alpha_1)}_{=0} + \underbrace{(x_i \alpha_i, x_2 \alpha_2)}_{=0} + ... + \underbrace{(x_i \alpha_i, x_i \alpha_i)}_{\ne 0} + ... + \underbrace{(x_i \alpha_i, x_n \alpha_n)}_{=0} = 0$;

$(x_i \alpha_i, x_i \alpha_i) = 0$, $\alpha_i = 0$;

Definition. An orthogonal system of elements of the Euclidean space L is called an *orthogonal basis* of this space if the smallest linear subspace that contains the orthogonal system coincides with the entire space L.

An orthonormal system of functions is said to be complete in some space if any element x from this space can be approximated as accurately as desired by a linear combination of basis vectors in the sense of the norm of the given space.

Control questions.

1. Define linear space. Give examples.
2. What linear space is called n-dimensional? Infinitely measurable?
3. Give the definition of a linear manifold.
4. Define norm and normed space.
5. Define scalar product and Euclidean space.
6. Write the Cauchy-Buniakovsky inequality in Euclidean space.
7. Give the definition of an orthonormal basis in Euclidean space.

Exercises.

1. Consider the set M_{mn} of all rectangular matrices of order $m \times n$ with scalar elements

$$(a_{ij})=\begin{pmatrix} a_{11}a_{12}...a_{1n} \\ a_{21}a_{22}...a_{2n} \\ .... \\ a_{m1}a_{m2}...a_{mn} \end{pmatrix}, \qquad (b_{ij})=\begin{pmatrix} b_{11}b_{12}...b_{1n} \\ b_{21}b_{22}...b_{2n} \\ .... \\ b_{m1}b_{m2}...b_{mn} \end{pmatrix}.$$

Defined in M_{mn} operations $(a_{ij})+(b_{ij})=(a_{ij}+b_{ij}),\ \lambda(a_{ij})=(\lambda a_{ij})$ Prove that M_{mn} is a linear space.

2. Let's put it in the normalized space $\| x-y \| = \rho(x,y)$. Check the fulfillment of the axioms of the metrics defined in this way.
3. Let $E_{[a,b]}$ – the linear space of all real functions defined on the segment $[a,b]$. Prove that space $C_{[a,b]}$ – continuous functions on a segment $[a,b]$. is a linear manifold.
4. In space C_l denote the scalar product by the formula $(x,y)=\int_a^b x(t)y(t)dt.$ Check the fulfillment of the scalar product axioms.

The solution.

Since operations on matrices are reduced to operations on numbers, the validity of the axioms of linear space is obvious..

1. 1. Let's check the fulfillment of the axioms of the metric, taking into account the fulfillment of the axioms of the norm.

   a) $\rho(x,y)=\|x-y\|\geq 0$, moreover $\rho(x,y)=0$ then, and only if $x\equiv y$.

   b) $\rho(x,y)=\|x-y\|=\|(-1)(y-x)\|=|(-1)|\,\|y-x\|=\rho(x,y)$.

   c) $\rho(x,y)=\|x-y\|=\|(x-z)+(z-y)\|\leq\|(x-z)\|+\|(z-y)\|=\rho(x,z)+\rho(z,y)$.

2. This follows from the fact known from mathematical analysis that a linear combination of two continuous on a segment $[a,b]$. of functions is a function that is continuous on this segment.

3. Let's check the fulfillment of the scalar product axioms. $(x,x)=\int_a^b x^2(t)dt$, moreover $(x,x)=0$ then, and only if $x(t)\equiv 0$.

   Indeed, if $x(t)\equiv 0$, than $(x,x)=\int_a^b 0dt=0$.

   If $(x,x)=\int_a^b x^2(t)dt=0$, then it follows that $x(t)\equiv 0$, when would $x(t)\neq 0$ at least at one point t_1, then by virtue of continuity the function $x(t)$ would be, for example, greater than zero and in some circle of the point t_1, and then equality would not be fulfilled $\int_a^b x^2(t)dt=0$.

   The fulfillment of the other three axioms of the scalar product follows from the properties of the definite integral.

Problems.

1. In space R^n and $C_{[a,b]}$ the operations of adding elements and multiplying an element by a number are introduced by formulas:

   а) $x+y=(x_1+y_1,...,x_n+y_n)$, $\lambda x=(\lambda x_1,...,\lambda x_n)$;

   б) $x+y=x(t)+y(t)$, $\lambda y=\lambda x(t)$;

Check the linearity of spaces R^n and $C_{[a,b]}$.

2. Show that in $C_{[0,\pi]}$ functions $1, \cos t, \cos^2 t$ are linearly independent, and functions $1, \cos 2t, \cos^2 t$ linearly dependent.
3. Prove that spaces $C_{[a,b]}, l_2$ infinitely measurable.
4. Show that the set of all polynomials of degree no higher than $m \in (m+1)-$ is a measurable linear manifold in $C_{[a,b]}$.
5. Show that in $C_{[a,b]}$ the set of all functions that satisfy the limits conditions $x(a)=\alpha, x(b)=\beta$, will be a linear manifold if and only if $\alpha=\beta=0$.
6. In spaces $R_n^0, C_{[a,b]}, C_l$ the norms of elements are determined by formulas: $\|x\|=\max|x_i|$; $\|x\|=\max|x(t)|$; $\|x\|=\int_a^b |x(t)|dt$ in accordance. Check the fulfillment of the axioms of the norms.
7. Prove the Cauchy-Buniakovsky inequality in Euclidean space.
8. In space l_2 the scalar product is defined by the formula $(x,y)=\sum_{k=1}^{\infty} x_k y_k$. Prove that the sequence $\sum_{k=1}^{\infty} x_k y_k$ – converges and the scalar product axiom holds. What does it look like in l_2 set of orthogonal elements.

LINEAR OPERATORS AND LINEAR FUNCTIONALS

Let there be two linear spaces L_1 and L_2. If each element has x space L_1 matched to a very specific element y from space L_2, then we say that the given operator $y = Ax$, which acts in space L_1 with values in L_2.

Definition. The operator A is called linear if for $\forall x_1, x_2 \in L_1$ and for any numbers α_1 and α_2 equality holds $A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$.

Example.

$$y=\int_a^b K(x,t)\varphi(t)dt.$$

The operator acts in space $C_{[a,b]}$. We will assume that K and φ – continuous Let's check the linearity condition:

$$\int_a^b K(x,t)(\alpha_1\varphi_1(t)+\alpha_2\varphi_2(t))dt=\alpha_1\cdot\int_a^b K(x,t)\varphi_1(t)dt+\alpha_2\cdot\int_a^b K(x,t)\varphi_2(t)dt=$$
$$=\alpha_1 A\varphi_1+\alpha_2 A\varphi_2.$$

Properties follow from the linearity of the operator:

1. $A(-x)=-Ax$;
2. $A(0)=A(x-x)=Ax-Ax=0$.

Definition. Line operator $y=Ax$ is called continuous at the point x_0 if from the convergence of any sequence $\{x_n\}\to x_0$ it follows that the corresponding sequence of operator values $\{Ax_n\}\to Ax_0$. Here, convergence is understood according to the norm of the given space, that is, if at $n\to\infty$ $\|x_n-x_0\|\to 0$, то $\|Ax_n-Ax_0\|\to 0$.

Theorem. If a linear operator is continuous at the point x_0, then it is also continuous at an arbitrary point in an arbitrary space.

Proof. Consider an arbitrary sequence $\{x_n\}\to x$ and we will show that $\{Ax_n\}\to Ax$. Consider an arbitrary element $(x_n-x+x_0)\to x_0$. Then due to continuity in x_0 $A(x_n-x+x_0)\to Ax_0$. Let's use linearity: $(Ax_n-Ax+Ax_0)\to Ax_0$, then $(Ax_n-Ax)\to 0,\quad Ax_n\to Ax$ (at $n\to\infty$).

Definition. . A linear operator that operates on a linear space with values in a linear space L_2 нis called bounded if such a number exists $M>0$, what for $\forall x$ $\|Ax\|\le M\|x\|$.

Theorem. In order for a linear operator to be continuous at any point x, it is necessary and sufficient that it is bounded.

Proof. Necessity. Given: $y=Ax$ – continuous operator. Let's prove the boundedness, that is, what $\exists M>0$, $\|Ax\|\le M\|x\|$. Assume the opposite, that the operator is unbounded. This means that for $\forall n\in N$ $\exists x_n$, $\|Ax_n\|>n\|x\|$, $n=1,2,3,...$.

Consider an element $z_n=\dfrac{x_n}{n\|x_n\|}$ and estimate according to the norm:

$\|z_n\|=\dfrac{1}{n\|x_n\|}\cdot\|x_n\|=\dfrac{1}{n}\to 0$, $n\to\infty$; $\|z_n-0\|\to 0$, $n\to\infty$, $z_n\to 0$. Then, by virtue of continuity, $\|Az_n\|\to 0$.

On the other hand $\|Az_n\|=\left\|A\dfrac{x_n}{n\|x_n\|}\right\|=\dfrac{1}{n\|x_n\|}\cdot\|Ax_n\|\ge\dfrac{n\|x_n\|}{n\|x_n\|}=1$, does not go to zero, $n\to\infty$. Got a contradiction.

Sufficiency. Given that the operator is bounded, i.e $\exists M>0$, $\|Ax\|\le M\|x\|$. We will prove that for $\forall x_n\to x$, $Ax_n\to Ax$.

$\|A(x_n-x)\|\le M\|x_n-x\|$; $0\le\|Ax_n-Ax\|\le M\|x_n-x\|\Rightarrow Ax_n\to Ax$.

The theorem is proved.

Operator rate.

Let us have a linear bounded operator: $\exists M>0$, $\|Ax\|\le M\|x\|$.

Definition. The smallest of the components M, which appears in the constraint condition is called the *norm* of the operator A and is denoted $\|A\|$, $\|Ax\|\le\|A\|\cdot\|x\|$.

In other words, a number $\|A\|$ is called the norm of the operator if for $\forall\varepsilon>0$, $\exists x_1$, $\|Ax_1\|>\|A-\varepsilon\|\cdot\|x_1\|$.

Example. We have a reflection $R^n\to R^1$,

$y=a_1x_1+a_2x_2+...+a_nx_n$.

$a=(a_1,a_2,...,a_n)$ – fixed vector.

$x=(x_1,x_2,...,x_n)$ – variable vector.

Let's find the norm of the operator

$$\|Ax\|=|Ax|=\underbrace{\left|\sum_{i=1}^{n}a_ix_i\right|\le\sqrt{\sum_{i=1}^{n}a_i^2}\cdot\sqrt{\sum_{i=1}^{n}x_i^2}}=\sqrt{\sum_{i=1}^{n}a_i^2}\cdot\|x\|,$$

$$\|A\|\le\sqrt{\sum_{i=1}^{n}a_i^2}.$$

Let's show what it really is $\|A\|$ equal to the right side. For this, it is enough to find the element x so that the inequality would turn into equality. Consider an element $x=\left(\frac{a_1}{\sqrt{\sum_{i=1}^{n}a_i^2}},\frac{a_2}{\sqrt{\sum_{i=1}^{n}a_i^2}},...,\frac{a_n}{\sqrt{\sum_{i=1}^{n}a_i^2}}\right)$, than:

$$\|Ax\|=|Ax|=\frac{a_1^2}{\sqrt{\sum_{i=1}^{n}a_i^2}}+...+\frac{a_n^2}{\sqrt{\sum_{i=1}^{n}a_i^2}}=\sqrt{\sum_{i=1}^{n}a_i^2}.$$

***Control questions*.**

:

1. Give the definition of a linear operator.
2. Define the continuity of a linear operator.
3. Which operator is called limited?
4. What is called the norm of the operator?
5. Give the definition of a linear functional. Give examples.
6. What is the general appearance of linear functionals in spaces R^n and $C_{[a,b]}$?
7. Which operator is called reversible?

Exercises.

1. Consider in the space of continuous functions on a line segment $[a,b]$. operator defined by the formula $\varphi(s)=\int_a^b k(s,t)\gamma(t)dt,$ where

$k(s,t)$ – some fixed continuous function of two variables. Prove the linearity of this operator.

2. Show that the differentiation operator $f(t) = f'(t)$, which operates in the subspace of continuous functions, is linear but not continuous.
3. To prove that in space R^n any linear functional can be represented by the formula $y = \sum_{k=1}^{n} a_k x_k$, де $a = (a_1, a_2 ... a_n)$ a well-defined vector, a $x = (x_1, x_2 ... x_n)$ – arbitrary vector.
4. Find the norm of the operator acting from space R^n in space R^1 and which is defined by equality $y = Ax = \sum_{k=1}^{n} a_k x_k$.

<u>The solution.</u>

1. We have: $A(\alpha x_1 + \beta x_2) = S\int_a^b k(s,t)(\alpha x_1(t) + \beta x_2(t))dt =$
$= \alpha \int_a^b k(s,t) x_1(t)dt + \beta \int_a^b k(s,tx_2(t)dt = \alpha Ax_1 + \beta Ax_2$, which proves the linearity of the operator.
2. The linearity of this operator is obvious. The fact that this operator is not continuous follows, for example, from the fact that the sequence $\varphi_n(t) = \frac{\sin(nt)}{n}$ converges to zero (in the metric $C_{[a,b]}$), and sequence $\varphi_n'(t) = \cos nt$ not.
3. Let's express x in terms of unit vectors $\vec{e}_i$ according to the formula $x = x_1 e_1 + x_2 e_2 + ... + x_n e_n$. Then, using the linearity of the functional, we get
$Ax = A(x_1 e_1 + ... + x_n e_n) = x_1 Ae_1 + ... + x_n An_n = a_1 x_1 + ... + a_n x_n$, which had to be proved.

4. We have that $\|Ax\| = |Ax| \left|\sum_{k=1}^{n} a_k x_k\right| \le \sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} x_k^2} = \sqrt{\sum_{k=1}^{n} a_k^2}\,\|x\|$.

According to the definition of the norm of the operator, it follows that $\|A\| = \sqrt{\sum_{k=1}^{n} a_k^2}$. Consider an element $x^* = \left(\frac{a_1}{\sqrt{\sum_{k=1}^{n} a_n^2}}, \dots, \frac{a_n}{\sqrt{\sum_{k=1}^{n} a_n^2}}\right)$. As $\| Ax^* \| = | Ax | = \sqrt{\sum_{k=1}^{n} a_k^2}$, то $\|A\| = \sqrt{\sum_{k=1}^{n} a_k^2}$.

Problems.

1. Prove that when the linear operator A is continuous at one point x_0 of a linear normed space R, then it is continuous in R.
2. Prove that any bounded linear operator is continuous.
3. Establish the general appearance of the linear functional in space l_2.
4. Functionals are given
   - $Ax = x(t_0),\ x(t) \in C_{[a,b]},\ t_0$ – fixed point from the segment $[a,b]$.
   - $Ax = x_0(t)x(t),\ x(t) \in C_{[a,b]},\ x_0(t)$ – fixed point from the segment $C_{[a,b]}$.
   - $Ax = \int_a^b x(t)dt,\ x(t) \in C_{[a,b]}$
   - $Ax = x,\ x = (x_1, \dots, x_n) \in R^n$

   Examine these functions for linearity, boundedness and find their norms.
5. Will the operator be linear $Ax = u + x(u),\ 0 \le u \le 1$?
6. Will the operator be linear p $Ax = u + x(0),\ 0 \le u \le 1$? Will it be a continuous operator with $C_{[0,1]}$ in $C_{[0,1]}$?

LINEAR ADDITIVE OPERATORS

In the theory of function approximations, a fundamental role is played by the Weierstrass theorem, which states that any continuous function $f(x)$ on the segment $[a,b]$. can be accurately approximated by an algebraic polynomial.

If the function is sufficiently smooth (has many derivatives) and the residual term of the Taylor formula goes to zero, then the Weierstrass theorem is obvious. The Taylor polynomial will act as a polynomial in this case.

However, a function can have derivatives of all orders and cannot be approximated by a Taylor polynomial.

In addition, you can imagine a function that is continuous and does not have a derivative at any point. It turns out that such a function can also be approximated by an algebraic polynomial. Let's prove the Weierstrass theorem using linear positive operators.

Consider a linear operator $L(f,x)$, which is given in the space of continuous functions $C_{[a,b]}$, so that it depends on the function f and from x. An example of such an operator can be Bernstein polynomials $B_n(f,x)=\sum_{k=0}^{n} f(\frac{k}{n})C_n^k x^k(1-x)^{n-k}$.

<u>Definition.</u> Line operator $L(f,x)$ is called positive if it belongs to every integral function $f(x)$ matches the integral value of the operator. The Bernstein operator will be such an operator for functions f, which are set at $[0,1]$.

It is clear that when $f_2 \ge f_1$, то $L(f_2,x) \ge L(f_1,x)$. Indeed, $f_2 - f_1 \ge 0$, then $L(f_2,x) - L(f_1,x) \ge 0$, $L(f_2 - f_1,x) \ge 0$.

<u>Theorem (Korovkin).</u> Let us have a sequence of linear positive operators $L_n(f,x)$, which are given in the space of continuous functions

$C_{[a,b]}$ and conditions are met:

1. $L_n(1,x) = 1 + \alpha_n(x)$,
2. $L_n(t,x) = x + \beta_n(x)$,
3. $L_n(t^2,x) = x^2 + \gamma_n(x)$,

where α_n, $\beta_{n,}, \gamma_n$ – are infinitely small, which uniformly go to zero along all x, at $n \to \infty$. Then the sequence of linear operators leads uniformly to a continuous function $f(x)$.

Proof. Because $f(x)$ continuous on $[a,b]$, then according to the first theorem of Weierstrass, it will be bounded, i.e:

$-M \le f(x) \le M$,

$-2M \le f(t) - f(x) \le 2M$. (1)

According to Cantor's theorem, the function $f(x)$ will be uniformly continuous on $[a,b]$. This means that for $\forall \varepsilon > 0$, $\exists \delta > 0$, $|t - x| < \delta$:

$|f(t) - f(x)| < \varepsilon$,

$-\varepsilon < f(t) - f(x) < \varepsilon$. (2)

From inequalities (1) and (2) the following inequality follows:

$$-\varepsilon - \frac{2M(t-x)^2}{\delta^2} < f(t) - f(x) < \varepsilon + \frac{2M(t-x)^2}{\delta^2}. \quad (3)$$

It is necessary to analyze two cases when $|t - x| < \delta$ and $|t - x| \ge \delta$. Let's act on inequality (3) operator and consider that it is positive and linear.

$$-\varepsilon L_n(1,x) - \frac{2M}{\delta^2} L_n((t-x)^2, x) < L_n(f,x) - f(x) L_n(1,x) < \varepsilon L_n(1,x) + \frac{2M}{\delta^2} L_n((t-x)^2, x) \quad (4)$$

We have: $L_n((t-x)^2, x) = L_n(t^2, x) - 2x L_n(t,x) + x^2 L_n(1,x) =$

$= x^2 + \gamma_n(x) - 2x(x + \beta_n(x)) + x^2(1 + \alpha_n(x)) = \gamma_n(x) - 2x\beta_n(x) + x^2 \alpha_n(x) = \theta_n(x) \to 0$,

at $n \to \infty$.

Then inequality (4) takes the form:

$$-\varepsilon(1 + \alpha_n(x)) - \frac{2M}{\delta^2}\theta_n(x) < L_n(f,x) - f(x)(1 + \alpha_n(x)) < \varepsilon(1 + \alpha_n(x)) + \frac{2M}{\delta^2}\theta_n(x).$$

With a sufficiently large natural M, we can achieve that the inequality holds:

$$-2\varepsilon < L_n(f,x) - f(x) < 2\varepsilon,$$

$$\left|L_n(f,x) - f(x)\right| < 2\varepsilon.$$

And this proves that the sequence of linear positive operators leads to $f(x)$ evenly (because the inequality holds for everyone x).

The theorem is proved.

<u>Bernstein's theorem.</u> Let us have a function $f(x)$, which is continuous on the segment $[0,1]$. Then the Bernstein polynomials $B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) C_n^k x^k (1-x)^{n-k}$ heading to $f(x)$ evenly.

<u>Proof.</u> For the proof, it is enough to make sure that the conditions of Korovkin's theorem are fulfilled for Bernstein polynomials.

Indeed, this operator is positive and linear. In addition, we have:

1. $B_n(1,x) = \sum_{k=0}^{n} 1 \cdot \frac{n(n-1)...(n-k+1)}{k!} x^k (1-x)^{n-k} = (x+(1-x))^n = 1 = 1+0$, $\alpha_n(x) = 0$.

The first condition is fulfilled.

2. $B_n(t,x) = \sum_{k=0}^{n} \frac{k}{n} C_n^k x^k (1-x)^{n-k}$,

$$\frac{k}{n} C_n^k = \frac{k}{n} \cdot \frac{n(n-1)...(n-k+1)}{k!} = \frac{(n-1)...(n-k+1)}{(k-1)!} = C_{n-1}^{k-1}.$$

We will have:

$$B_n(t,x) = \sum_{k=1}^{n} C_{n-1}^{k-1} x^k \left(1-x\right)^{n-k} = x \sum_{k=1}^{n} C_{n-1}^{k-1} x^{k-1} \left(1-x\right)^{n-1-(k-1)} = x\left(x+(1-x)\right)^{n-1} = x = x+0,$$

$\beta_n = 0$.

3. $B_n(t^2,x) = x^2 + \gamma_n(x)$,

$$B_n(t^2,x) = \sum_{k=0}^{n} \frac{k^2}{n^2} C_n^k x^k (1-x)^{n-k} = \sum_{k=1}^{n} \frac{k}{n} C_{n-1}^{k-1} x^k (1-x)^{n-k} = \frac{n-1}{n} \sum_{k=1}^{n} \frac{k-1}{n-1} C_{n-1}^{k-1} x^k (1-x)^{n-k} +$$

$$+ \frac{1}{n} \sum_{k=1}^{n} C_{n-1}^{k-1} x^k (1-x)^{n-k} = \frac{n-1}{n} \sum_{k=2}^{n} C_{n-2}^{k-2} x^k (1-x)^{n-k} + \frac{x}{n} \sum_{k=1}^{n} C_{n-1}^{k-1} x^{k-1} (1-x)^{n-k} =$$

$$=\frac{n-1}{n}x^2\sum_{k=2}^{n}C_{n-2}^{k-2}x^{k-2}(1-x)^{n-2-(k-2)}+\frac{x}{n}\sum_{k=1}^{n}C_{n-1}^{k-1}x^{k-1}(1-x)^{n-1-(k-1)}=\frac{n-1}{n}x^2+\frac{x}{n}=x^2-\frac{x^2}{n}+\frac{x}{n}=$$

$= x^2+\gamma_n(x),\ \gamma_n(x)$ – infinitely small.

All conditions of Korovkin's theorem are fulfilled. Therefore, the Bernstein polynomials converge uniformly to the function $f(x)$, which is set to $[0,1]$.

<u>Weierstrass theorem.</u> For any function $f(x)$ continuous on an arbitrary segment $[a,b]$ there is a sequence of algebraic polynomials $P_n(x)$, which go uniformly towards the function $f(x)$.

<u>Proof.</u> Let's make a replacement $x=a+(b-a)y$, if $x=a$, *y=0*,

$x=b$, *y=1*.

Then the function $f(a+(b-a)y)=\varphi(y)$ set on the segment $[0,1]$. This function $\varphi(y)$, which is already set to $[0,1]$, we can approximate the Bernstein polynomial as accurately as desired $P_n(y)$:

$|\varphi(y)-P_n(y)|<\varepsilon$.

Let's go back to the old variablesої *x*: $y=\frac{x-a}{b-a}$. As a result of this substitution, the function $\varphi(y)$ will transform в $f(x)$, $x\in[a,b]$, and $P_n(y)$ will be an algebraic polynomial $P_n^*(x)$. We will have that $|f(x)-P_n^*(x)|<\varepsilon$.

The theorem is proved.

<u>Control questions</u>.

1. Which operator on the space of continuous functions is called positive?
2. Prove the linearity and positiveness of the Bernstein and Valle Poisson operators.

3. Prove Korovkin’s theorem on the convergence of a sequence of linear positive operators to a continuous function $f(x)$.
4. Prove Bernstein's theorem..
5. Prove the Weierstrass theorem on the convergence of a sequence of algebraic polynomials to a continuous one on a segment $[a,b]$ function $f(x)$.

TEST

<u>Sample test.</u>

1. $f(x)=\frac{1}{2}\ln x,\ 1\le x\le\infty$. Prove that this mapping is compressive. However, equation $x=f(x)$ has no valid roots. Why?
2. Prove that a closed unit square is compact.
3. In space R_0^n the norm of the element is entered according to the formula $\|x\|=\max|x_i|$. Check the fulfillment of the norm axiom.
4. Find the norm of the functional $Ax=\int_{-4}^{4}(t^2-1)x(t)dt,\quad x(t)\in C_{[a,b]}$
5. Using the Heine-Borel lemma, prove the first Bolzano-Cauchy theorem.

<u>The solution.</u>

1. This reflection is not a reflection in itself. (Why?). Therefore, one of the conditions of Banach's theorem is not fulfilled.
2. According to the Bolzano-Weierstrass theorem on a bounded sequence in space R_2 a convergent subsequence can be distinguished. Due to the closedness of the unit square, the limit of the subsequence will belong to this square.
3. We have:

a). $\|x\|\geq 0$, moreover $\|x\|=0$ if and only if all coordinates of the vector $x=(x_1,...,x_n)$ levels are zero;

b). $\|\lambda x\|=\max\limits_i |\lambda x_i|=|\lambda|\max\limits_i |x_i|=|\lambda|\|x\|$;

c). $|x_l+y_i|\leq|x_i|+|y_i|\leq\max\limits_i |x_i|+\max\limits_i |y_i|=\|x\|+\|y\|$.

This inequality holds for all *i*, in particular, when the maximum of all *i* stands in the left part, i.e $\|x+y\|\leq\|x\|+\|y\|$.

4. $\|Ax\|=|Ax|=\left|\int\limits_{-2}^{2}(t^2-1)x(t)dt\right|\leq\int\limits_{-2}^{2}\left|t^2-1\right|\left|x(t)\right|dt\leq\int\limits_{-2}^{2}\left|t^2-1\right|\left\|x(t)\right\|dt=$

$=\|x\|\int\limits_{-2}^{2}\left|t^2-1\right|dt\ =\|x\|\cdot 2\left(\int\limits_{0}^{1}(1-t^2)dt+\int\limits_{1}^{2}(t^2-1)dt\right)=4\|x\|$. So, $\|A\|\leq 4$.

Taking now as a function $x(t)$ somewhat smoothed function $sign(t^2-1)$, let's make sure that in fact $\|A\|=4$.

5. Suppose, reasoning from the opposite, that at no point of the segment $[a,b]$ function $f(x)$ does not change to zero, i.e. preserves the sign at all internal points of the segment $[a,b]$. Due to the continuity of the function $f(x)$ з will preserve the sign in some neighborhood of the point *x*. This system of circles forms an infinite open coverage of the segment $[a,b]$. According to the Heine-Borel lemma, a finite subcover can be isolated from this infinite covering, in each of whose intervals the function preserves its sign. And this contradicts the fact that at the ends of the segment $[a,b]$ the function has different signs.

PROGRAM QUESTIONS FOR THE EXAM

1. From the history of the emergence and development of functional analysis. Subject of functional analysis.
2. Metric spaces. Examples.
3. Sequences in metric spaces. Sequence limit. Properties.
4. Convergence in metric spaces. Convergence in R^n.
5. Convergence in space l_2.
6. Convergence in space $C_{[a,b]}$.
7. Circle points in metric space. Open and closed sets, the connection between them.
8. Representation of metric spaces. Examples of continuous mappings. Criterion of continuity.
9. Connectivity of a set and its preservation under continuous mapping.
10. Completeness of metric spaces. Completeness of a closed subspace of a complete space.
11. Nested Spheres Theorem.
12. The completeness $C_{[a,b]}$.
13. The completeness R^n.
14. The completeness l_2.
15. The principle of compressive mappings, Banach's theorem.
16. Application of Banach's theorem to the solution of type equations $x = f(x)$, $F(x) = 0$.
17. Application of Banach's theorem to solving n linear equations with n unknowns.
18. Application of Banach's theorem to prove the existence and unity of the solution of the Cauchy problem of the differential equation $y' = f(x, y)$.

19. Application of Banach's theorem to prove the existence and unity of the solution of the Cauchy problem for a normal system of differential equations.
20. Solving non-linear Fredholm integral equations of the second kind.
21. Solving Hammerstein's nonlinear integral equations.
22. Solving nonlinear integral equations of Voltaire of the second knd.
23. Compactness. An example of a space in which the Bolzano-Weierstrass theorem does not hold.
24. Theorem on the closedness and boundedness of a compact set.
25. Compact in R^n .
26. The compactness of the image with continuous mapping.
27. Continuity of the inverse mapping of the compact.
28. Cantor's theorem on a compact.
29. Numerical functions on the compact (I and II theorems of Weierstrass).
30. Linear spaces. Examples. Infinitely measurable spaces.
31. Standard spaces. Examples. The metric generated by the norm.
32. The Cauchy-Buniakovsky inequality in an arbitrary linear space.
33. Euclidean spaces. Orthogonality.
34. Linear operators. Continuity and limitation.
35. The norm of the linear operator. Examples.
36. Linear functionals. General view of a linear functional in some spaces.
37. Linear positive operators. Korovkin's theorem.
38. Bernstein's theorem.
39. Weierstrass theorem on approximation of continuous functions by algebraic polynomials.

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